FROM CHEREDNIK ALGEBRAS TO KNOT HOMOLOGY VIA CUSPIDAL $\mathcal{D}$-MODULES

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ABSTRACT. We show that the triply-graded Khovanov-Rozansky homology of the torus knot $T_{m,n}$ can be recovered from the finite-dimensional representation $\mathrm{L}_{\frac{m}{n}}$ of the rational Cherednik algebra at slope $\frac{m}{n}$, endowed with the Hodge filtration coming from the cuspidal character $\mathcal{D}$-module $\mathbf{N}_{\frac{m}{n}}$. Our approach involves expressing the associated graded of $\mathbf{N}_{\frac{m}{n}}$ in terms of a DG module closely related to the action of the elliptic Hall algebra on the equivariant K-theory of the Hilbert scheme of points on $\mathbb{C}^2$, thereby proving the rational master conjecture. As a corollary, we identify the Hodge filtration with the inductive and algebraic filtrations on $\mathrm{L}_{\frac{m}{n}}$.

CONTENTS

1. INTRODUCTION

1.1. **Rational Cherednik algebras and link homology.** Recent years have seen extraordinary connections between seemingly unrelated mathematical objects across different fields, indexed by pairs of coprime natural numbers m, n. Topologically, there is the (m,n)-torus knot $T_{m,n}$, which winds m times around a circle in the interior of the torus, and n times around its axis of rotational symmetry. Surprisingly, the **triply graded** Khovanov-Rozansky homology [Kho07] of $T_{m,n}$ can be captured by the following representation-theoretic object.

For $n > 0$, we let $\mathfrak{h}$ denote the $(n-1)$-dimensional standard representation of the symmetric group S_n and $\mathcal{D}(\mathfrak{h})$ be the ring of differential operators on $\mathfrak{h}$. The rational Cherednik algebra $\mathrm{H}_c(S_n, \mathfrak{h})$ [EG02], also known as the rational degeneration of the double affine Hecke algebra, is a deformation of $\mathcal{D}(\mathfrak{h})\#S_n$ at a complex parameter c. It turns out that only when $c = \frac{m}{n}$ for coprime m, n does $\mathrm{H}_c(S_n, \mathfrak{h})$ have finite-dimensional representations; in that case, there is an unique irreducible one, usually denoted by $\mathrm{L}_{\frac{m}{n}}$ [BEG03]. There is a Euler grading on $\mathrm{L}_{\frac{m}{n}}$ induced by the action of the Euler field $\mathrm{h}_{\frac{m}{n}}$, which assigns every homogeneous polynomial its degree minus the delta invariant of $T_{m,n}$. This grading induces a direct sum decomposition $\mathrm{L}_{\frac{m}{n}} = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{L}_{\frac{m}{n}}(\ell)$.

On the other hand, the Khovanov-Rozansky homology of a knot is defined by taking the Hochschild homology of the associated Rouquier complex built upon Soergel bimodules. We refer the readers to [Kho07] for a detailed definition. Gorsky, Oblomkov, Rasmussen and Shende made the following remarkable observation:

Conjecture 1.1. *([GORS14, Conjecture 1.2]) For positive coprime integers m, n, there exists a filtration F on $\mathrm{L}_{\frac{m}{n}}$ compatible with the order filtration on $\mathrm{H}_{\frac{m}{n}}(S_n, \mathfrak{h})$ and the $\mathrm{h}_{\frac{m}{n}}$-grading such that there is an isomorphism of triply graded vector spaces:*

$$\bigoplus_{i,j,k} \mathrm{HHH}^{i,j,k}(T_{m,n}) \cong \bigoplus_i \mathrm{Hom}_{S_n}\big(\wedge^i \mathfrak{h}, \bigoplus_{j,k} \mathrm{gr}^F_j\big(\mathrm{L}_{\frac{m}{n}}(k)\big)\big). \tag{1}$$

such that the following gradings are matched:

$$\text{Hochschild homological } a\text{-grading} \leftrightarrow \text{degree of } \wedge^\bullet \mathfrak{h}$$
$$\text{internal } q\text{-grading} \leftrightarrow \mathrm{h}_{\frac{m}{n}}\text{-grading}$$
$$\text{usual homological } t\text{-grading} \leftrightarrow \text{filtration on } \mathrm{L}_{\frac{m}{n}}$$

A bigraded isomorphism is established in [GORS14] without the information of the t-grading. For any coprime pair (m, n), $m > n > 0$, we define a Hodge filtration on $\mathrm{L}_{\frac{m}{n}}$ and show that:

Theorem A. *(Theorem 5.15) When $m > n$ such that $(m, n) = 1$, Conjecture 1.1 holds with respect to the Hodge filtration.*

For the cases when $m > 0$, see Corollary 1.3.

1.2. **Hilbert schemes and the rational master conjecture.** Write $c = \frac{m}{n}$ for positive coprime integers m, n. Our approach is to construct a coherent sheaf $\mathcal{F}_c$ on Hilb^n, the Hilbert scheme of n points on $\mathbb{C}^2$. This sheaf is used to compute the graded dimensions of both sides of isomorphism (1). Let $K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n)$ denote the equivariant K-theory of Hilb^n where the $\mathbb{C}^* \times \mathbb{C}^*$-action is induced by its scaling action on $\mathbb{C}^2$. The generalized McKay correspondence proved by Haiman [Hai01] implies an isomorphism

$$K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n) \cong \{f \in \mathbb{C}(q,t)[z_1, \cdots, z_n]^{S_n} | f((1-q)(1-t)z) \text{ has coefficients in } \mathbb{C}[q^\pm, t^\pm]\}. \tag{2}$$

The variables t, q correspond to the j, k gradings in (1) respectively. The i grading comes from the coefficient of the Schur polynomial labeled by the S_n-representation $\wedge^i \mathfrak{h}$, known as the hooks.

Beyond the hooks, Gorsky and Neguţ conjectured that the bigraded Frobenius character of $\mathrm{L}_{\frac{m}{n}}$ can be expressed via the action of a generator $P_{m,n}$ of the elliptic Hall algebra (EHA) on its polynomial representation, identified with $\bigoplus_n K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n) \otimes_{\mathbb{C}[q^\pm,t^\pm]} \mathbb{C}(q,t)$.

Conjecture 1.2. [GN15, Conjecture 5.5] *For positive coprime integers m, n, the bigraded Frobenius character of L_c with respect to an appropriate filtration and the $\mathrm{h}_{\frac{m}{n}}$-grading is given by*

$$\mathrm{ch}_{S_n\times\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{L}_{\frac{m}{n}}) = (P_{m,n} \cdot 1)(q, q^{-1}t).$$

Theorem B. *(Theorem 5.13) When $m > n$ such that $(m, n) = 1$, Conjecture 1.2 holds with respect to the Hodge filtration.*

For the cases when $m > 0$, also see Corollary 1.3. In the case of $m = n + 1$, $\mathrm{L}_{\frac{n+1}{n}}$ is isomorphic to the *diagonal harmonics* and the EHA generator action can be identified with the action of the nabla operator [BG98]. Therefore, Conjecture 1.2 may be viwed as a rational generalization of the master conjecture [GH96] proved by Haiman [Hai02]. Haiman's proof of the master conjecture (and those of the $n!$, $(n+1)^{n-1}$, and Macdonald positivity conjectures [Hai01]) is based on the geometry of the Hilbert scheme of points on the plane. One important input of Haiman's proofs of these conjectures is the introduction of the Procesi bundle of rank $n!$. Our desired sheaf $\mathcal{F}_{\frac{n+1}{n}}$ on Hilb^n is exactly the extension by 0 of the restriction of the Procesi bundle to the punctual Hilbert scheme, which is the zero fiber of the Hilbert-Chow map. Computing the bigraded Frobenius character of fibers of these sheaves at fixed points will lead to our proof of the GN conjecture.

The computation of the Khovanov-Rozansky homology for torus links had long been a challenging open problem, which was finally addressed by Elias, Hogancamp, and others ([Hog17, EH19, Hog18]) through recursive methods. The culmination of this work is a shuffle conjecture style formula [Mel22]. From this formula, it follows that the Euler characteristic of $\mathrm{HHH}(T_{m,n})$ equals the knot superpolynomial of $T_{m,n}$, interpreted as the matrix coefficient of $\sum[\wedge^i\mathfrak{h}]$ in $P_{m,n}\cdot 1$, as observed in [GN15]. Therefore, Theorem B implies Theorem A.

1.3. **DG flag commuting varieties.** To construct the coherent sheaf $\mathcal{F}_c$ on Hilb^n, we consider the cuspidal mirabolic D-module $\overline{\mathbf{N}}_c$ on $\mathfrak{sl}_n\times\mathbb{C}^n$ of central character c, closely related to the cuspidal character sheaf in the sense of Lusztig [Lus86]. A result of Calaque, Enriques and Etingof [CEE09] implies that $(\mathrm{L}_c)^{S_n}$ is the *quantum Hamiltonian reduction* of $\overline{\mathbf{N}}_c$. The desired coherent sheaf $\mathcal{F}_c$ on Hilb^n is obtained via the descent of the associated graded of $\overline{\mathbf{N}}_c$ with respect to the Hodge filtration within the framwork of Saito's theory of mixed Hodge modules [Sai90].

New geometry occurs in our study of the associated graded of $\overline{\mathbf{N}}_c$, as we will explain in the rest of this subsection. First of all, as noted by Neguţ in [Neg15a], the action of the generator $P_{m,n}$ on $K^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2))$ is not visible using the classical Nakajima correspondence, and the notion of a flag Hilbert scheme turns out to be necessary. However, it is well-known that the naive flag Hilbert scheme is highly singular [Che98]. A remedy is to rather consider the DG flag Hilbert scheme $\mathrm{FHilb}^n_{\mathrm{dg}}(\mathbb{C}^2)$, defined independently in [Gin12] and [Neg15a]. In spite of its derived nature, $\mathrm{FHilb}^n_{\mathrm{dg}}(\mathbb{C}^2)$ is by definition a local complete intersection.

In [Gin12], Ginzburg studies the isospectral commuting variety by expressing the associated graded of the Harish-Chandra $\mathcal{D}$-module with respect to the Hodge filtration in terms of the DG flag commuting variety. In type A, this facilitates the study of the Procesi bundle from a different perspective, which leads to Gordon's new proof of the Macdonald positivity conjecture ([Gor12]).

Our setup parallels that of [Gin12] and we naturally obtain the *cuspidal DG module* $\mathcal{A}_c$. Motivated by the similar origins of $\mathcal{A}_c$ at $c=0$ and the DG flag commuting variety, I expect that $\mathcal{A}_0$, whose definition is *valid for all types*, gives the correct definition for the DG nilpotent flag commuting scheme.

We also define a Catalan DG module, encoding the information of the q,t-Catalan number. We show the pushforwards of these two DG modules to the commuting variety correspond to the same equivariant K-theory classes and we state the sheaf-theoretic identification as a conjecture (Conjecture 5.3). While this paper was under editing, Gorsky and Neguţ generously shared a copy of their preprint [GN24]. Their Conjecture 2.2 is a non-derived version of our Conjecture 5.3.

1.4. **Filtrations and future directions.** Regarding Conjecture 1.1, the authors of [GORS14] have proposed several filtrations: (a) The inductive filtration is defined inductively using the shift functor from $\mathrm{H}_{\frac{m}{n}}$-modules to $\mathrm{H}_{\frac{m}{n}+1}$-modules and the "flipping" isomorphism $(\mathrm{L}_{\frac{m}{n}})^{S_n}\cong(\mathrm{L}_{\frac{n}{m}})^{S_m}$. (b) The Chern filtration is defined using powers of Chern classes in the equivariant cohomology of the compactified Jacobian of the planar singular curve $y^m=x^n$. (c) The geometric filtration is defined in terms of the perverse filtration and the cohomological grading on the cohomology of a Hitchin fiber isomorphic to the compactified Jacobian as in (b). The Chern is shown to coincide with the perverse filtrations, resp. the inductive filtration, in [OY16], resp. in [Ma24].

Theorem C. *(Proposition 5.18) The Hodge filtration equals the inductive and algebraic filtrations on* L_c *when* $m>n$ *for coprime* m,n *and on* $\mathrm{L}_c^{S_n}$ *for all* $m>0$ *coprime to* n.

We note that the inductive filtration and the Hodge filtrations are defined on L_c for $m>n$ and on $\mathrm{L}_c^{S_n}$ for all $m>0$ while the algebraic filtration is well-defined on L_c for all $m>0$. As a corollary of Theorem C, we show that:

Corollary 1.3. *(Proposition 5.21) For all integers* $m>0$ *coprime to* n*, with respect to the algebraic filtration, Conjectures 1.1 and 1.2 hold.*

There are two natural generalizations of our setting: replacing $\mathrm{Hilb}^n(\mathbb{C}^2)$ by Gieseker varieties and allowing m,n to be non-coprime. The former is related to the study of representations of a quantized Gieseker moduli algebra ([EKLS21]) and the study of higher Catalan numbers and a finite shuffle conjecture ([GSV23]). The latter is related to the study of rational Cherednik representations of minimal support and torus link homology ([EGL15]). Despite the current absense of definitions of the inductive, algebraic and geometric filtrations in these settings, we believe that a notion of Hodge filtration is still available, and via a similar method, bigraded Frobenius characters can be computed and related to the corresponding link invariants or Catalan statistics.

On the other hand, it is conjectured that the stable envelopes on $\mathrm{Hilb}^n(\mathbb{C}^2)$ are closely related to Verma modules of Cherednik algebras [GN17, Conjecture 6.5]. In general, the connection between Verma modules over quantized symplectic resolutions and categorical stable envelopes is studied by Bezrukavnikov and Okounkov [BO] using positive characteristic technique. From a different perspective, suitable filtrations on Verma modules, satisfying properties such as compatibility with parabolic induction and restriction functors, can also validate such a program. We believe the Hodge filtration might be the correct candidate; however, it is neither clear nor has it been explored which explicit $\mathcal{D}$-modules correspond to the Verma modules. We plan to investigate these directions in future work.

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2. Cherednik algebras, mirabolic D-modules, Hilbert schemes

Fix a positive integer n. We work over $\mathbb{C}$ throughout, and set up the following notations:
$\overline{G} = \mathrm{GL}_n$ with Lie algebra $\bar{\mathfrak{g}} := \mathfrak{gl}_n$;
$G = \mathrm{SL}_n$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$;
T: a maximal torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, consisting of diagonal matrices;
$W = S_n$ is the Weyl group;
$S \subset W$: the set of reflections.

2.1. **Rational Cherednik algebras.** For any $c \in \mathbb{C}$, we define the rational Cherednik algebra $\mathrm{H}_c := \mathrm{H}_c(S_n, \mathfrak{h})$ to be the $\mathbb{C}$-algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and W with relations

$$[x, x'] = [y, y'] = 0, \quad wxw^{-1} = w(x), \quad wyw^{-1} = w(y)$$
$$[y, x] = x(y) - \sum_{s \in S} c\langle \alpha_s, y\rangle \langle \alpha_s^\vee, x\rangle s$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $w \in W$. Here α_s, resp. $\alpha_s^\vee$, is the root, resp. coroot, associated to s and $(-,-)$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

For an irreducible W-representation τ, we regard it as a $S(\mathfrak{h})\#W$-representation by requiring $\mathbb{C}[\mathfrak{h}^*]$ to act trivially and define the Verma module

$$M_c(\tau) = \mathrm{H}_c \otimes_{S(\mathfrak{h})\#W} \tau.$$

The Verma module $M_c(\tau)$ has a unique nonzero irreducible quotient $\mathrm{L}_c(\tau)$.

Example 2.1. *When $c = 0$, $\mathrm{H}_0 = \mathcal{D}(\mathfrak{h})\#W$ whose polynomial representation $\mathbb{C}[\mathfrak{h}] \cong M_0(\mathrm{triv})$ is irreducible.*

Theorem 2.2. *([BEG03, Theorem 1.2]) When $c = \frac{m}{n}$ for positive integer m coprime to n, the unique irreducible finite-dimensional representation of H_c is $\mathrm{L}_c(\mathrm{triv})$. Moreover, only when $c = \frac{m}{n}$ for integer m coprime to n does H_c have finite-dimensional representations.*

We will simply write $\mathrm{L}_c := \mathrm{L}_c(\mathrm{triv})$. Let $\mathcal{O}(\mathrm{H}_c)$ denote the BGG category $\mathcal{O}$ of the rational Cherednik algebra, which is a full subcategory of $\mathrm{H}_c{-}\mathrm{mod}$ whose objects are finitely generated over H_c such that the $S(\mathfrak{h})$ action is locally nilpotent ([GGOR03]). Then as τ varies over irreducible W-representations, the modules $\mathrm{L}_c(\tau)$ give a complete list of irreducible objects in $\mathcal{O}(\mathrm{H}_c)$.

Let $\mathrm{e} := \frac{1}{n!}\sum_{w\in W} w$ be the symmetrizing idempotent in H_c. Then inside H_c, one has $\mathrm{e}\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^W$ and $\mathrm{e}S(\mathfrak{h}) \cong S(\mathfrak{h})^W$. The spherical Cherednik algebra is defined by $\mathrm{A}_c := \mathrm{e}\mathrm{H}_c\mathrm{e}$. By [BE09, Theorem 4.1], for all c satisfying

$$c \notin \{\frac{a}{b} \in (-1,0) | a, b \in \mathbb{Z}, 2 \le b \le n\} \tag{3}$$

there is a Morita equivalence

$$\mathrm{H}_c{-}\mathrm{mod} \to \mathrm{A}_c{-}\mathrm{mod}, \quad M \mapsto \mathrm{e}M.$$

Denote the BGG category $\mathcal{O}$ of the spherical Cherednik algebra by $\mathcal{O}(\mathrm{A}_c)$.

Let R^+ be a chosen set of positive roots. Write $\delta = \prod_{\alpha\in R^+}\alpha$. Let $\mathfrak{h}_r = \{\delta \neq 0\}$ denote the regular locus of $\mathfrak{h}$, i.e., when the diagonals are pairwise distinct. The action of H_c on $\mathbb{C}[\mathfrak{h}] \cong \mathrm{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\#W} \mathbb{C}$ gives the Dunkl embedding of H_c into $\mathcal{D}(\mathfrak{h}_r)\#W$. In particular, assume $\{x_i\}$ is a basis of $\mathfrak{h}^*$ and $\{y_i\}$ is its dual basis of $\mathfrak{h}$. Let (ij) denote the reflection swapping i and j. Then the Dunkl operator associated to y_i is

$$D_{y_i,c} := \partial_{x_i} - c\sum_{s\in S}\frac{\alpha_s(x_i)}{\alpha_s}(1-s) = \partial_{x_i} - c\sum_{j\neq i}\frac{1-(ij)}{x_i - x_j}.$$

Localized at $\delta := \prod_{\alpha\in\Delta^+}\alpha$, we have that $(\mathrm{H}_c)_\delta \cong \mathcal{D}(\mathfrak{h}_r)\#W$. Consider the symmetry on $(\mathrm{H}_c)_\delta$ defined by sending

$$x \mapsto x, \quad D_{y_i,c} \mapsto D_{y_i,-c}, \quad w \mapsto \mathrm{sign}(w)w$$

which restricts to an isomorphism $\mathrm{A}_d \to \delta^{-1}\mathrm{A}_{-d-1}\delta$ ([GGS09, 5.6, 5.8]). This induces an equivalence of categories, which we will use later:

$$\Omega_d = \Omega_{-d-1}^{-1} : \mathrm{A}_d{-}\mathrm{mod} \cong \mathrm{A}_{-d-1}{-}\mathrm{mod}.$$

2.2. **Quantum Hamiltonian reduction.** For any smooth algebraic variety X, we denote the sheaf of differential operators on X by $\mathcal{D}_X$ and write $\mathcal{D}(X) := \Gamma(X, \mathcal{D}_X)$. We say a filtration of a coherent $\mathcal{D}_X$-module M is good if $\widetilde{\mathrm{gr}}(M)$ is a coherent $\mathcal{O}_{T^*X}$-module. We denote the singular support of M by $SS(M)$.

Let $V = \mathbb{C}^n$ on which $\overline{G}$ act by left multiplication and $\mathfrak{G} = \mathfrak{g} \times V$. Let $\tau : \overline{\mathfrak{g}} \to \mathcal{D}(\mathfrak{G})$ denote the embedding induced by differentiating the diagonal action of $\overline{G}$ on $\mathfrak{G}$. Our convention is that for any $f \in \mathcal{O}(\mathfrak{G})$ and $g \in \overline{G}$,

$$(g \cdot f)(x) = f(g^{-1}\cdot x). \tag{4}$$

In particular, let $\mathbf{1} \in \overline{\mathfrak{g}}$ denote the identity matrix. Then $\tau(\mathbf{1}) = -\sum_{i=1}^n v_i\partial_{v_i}$.

Define the shifted embedding τ_d by a constant $d \in \mathbb{C}$ by (later we will take d to be $-c$)

$$\tau_d : \overline{\mathfrak{g}} \to \mathcal{D}(\mathfrak{G}), \quad x \to \tau(x) - d\cdot\mathrm{tr}(x).$$

Throughout, we identify $\mathfrak{g} \cong \mathfrak{g}^*$ under the Killing form. Inside $T^*\mathfrak{G} \cong \mathfrak{g}\times\mathfrak{g}\times V\times V^*$, we define a Lagrangian subvariety:

$$\Lambda := \{(X, Y, i, j) \in \mathfrak{g}\times\mathscr{N}\times V\times V^* | [X,Y] + ij = 0\}.$$

Definition 2.3. [FG10, GG06] *A finitely generated $\mathcal{D}_{\mathfrak{G}}$-module is **mirabolic** if it is locally finite as a $\tau(\overline{\mathfrak{g}})$-module and its singular support is contained in Λ.*

Denote the category of mirabolic $\mathcal{D}_{\mathfrak{G}}$-modules by $\mathscr{C}(\mathfrak{G})$. Fix a nonzero element vol $\in \bigwedge^n V^*$. Consider the regular function on $\mathfrak{G}$ defined by [BFG06, (5.3.2)]:

$$s(x,v) = \langle \mathrm{vol}, v \wedge xv \wedge \cdots \wedge x^{n-1}v\rangle. \tag{5}$$

Let $\mathfrak{G}_{\mathrm{cyc}} := \{(x,v) \in \mathfrak{G} | s(x,v) \neq 0\}$, which is equivalently the locus when v is a cyclic vector for x, i.e., $\mathbb{C}[x]v = V$. It is well-known that the composition of the projection $\mathfrak{G}_{\mathrm{cyc}} \to \mathfrak{g}$ and the Chevalley map $\mathfrak{g} \to \mathfrak{h}/W$ is a principal $\overline{G}$-bundle, such that the diagonal $\overline{G}$-action on $\mathfrak{G}_{\mathrm{cyc}}$ acts freely on the fibers. Therefore we have an isomorphism

$$\iota : \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\overline{G}} \cong \mathbb{C}[\mathfrak{h}]^W. \tag{6}$$

Note that $s^{-d} \in \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\tau_d(\overline{\mathfrak{g}})}$. In fact $\mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^{\tau_d(\overline{\mathfrak{g}})} = \mathbb{C}[\mathfrak{G}_{\mathrm{cyc}}]^G s^{-d}$. Moreover,

Theorem 2.4. *(*[GG06, Theorem 1.3.1], [GGS09, Theorem 8.1] *The radial part map*

$$\mathcal{D}(\mathfrak{G})^{\overline{G}} \to \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \Big[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \iota\big(s^d u\big(s^{-d}\iota^{-1}(f)\big)\big)\Big]$$

defines an isomorphism

$$\mathscr{H}_d : (\mathcal{D}(\mathfrak{G})/\mathcal{D}(\mathfrak{G})\tau_d(\overline{\mathfrak{g}}))^{\overline{G}} \cong \mathrm{A}_{d-1}, \tag{7}$$

which induces the quantum Hamiltonian reduction functor

$$\mathbb{H}_d : \mathscr{C}(\mathfrak{G}) \to \mathcal{O}(\mathrm{A}_{d-1}), \quad \mathbf{M} \mapsto \Gamma(\mathfrak{G}, \mathbf{M})^{\tau_d(\overline{\mathfrak{g}})}$$

such that

$$\mathscr{C}(\mathfrak{G})/\mathrm{Ker}(\mathbb{H}_d) \cong \mathcal{O}(\mathrm{A}_{d-1}).$$

Remark 2.5. *The "-1" factor in the subscript of* A_{d-1} *is consistent with the classical result of Harish-Chandra that*

$$\mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}/W), \quad u \mapsto \Big[\mathbb{C}[\mathfrak{h}/W] \ni f \mapsto \delta(u|_{\mathbb{C}[\mathfrak{h}/W]})\big(\delta^{-1}f\big)\Big]$$

induces an isomorphism $\big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathrm{ad}(\mathfrak{g})\big)^G \cong D(\mathfrak{h})^W$. *Here* δ *is the product of all positive roots and the restriction* $u|_{\mathbb{C}[\mathfrak{h}/W]}$ *is taken with respect to the Chevalley isomorphism* $C[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$.

2.3. Cuspidal mirabolic $\mathcal{D}$-modules.

2.3.1. *Functors on $\mathcal{D}$-modules.* Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We write $f_\dagger, f_!, f^\dagger$ to denote the derived pushforward, proper pushforward and pullback functors of D-modules respectively.

When f is a locally closed embedding, we define the minimal extension functor $f_{!*}$ to be the image under the canonical morphism $f_! \to f_\dagger$. We say a local system on a locally closed subset $U \xrightarrow{i} X$ is clean if its minimal extension coincides with the extensions using $f_\dagger$ or $f_!$.

2.3.2. *Cuspidal local systems.* Let $\mathscr{N}$ be the nilpotent cone in $\mathfrak{g}$ consisting of nilpotent matrices. Then the 0 fiber of the fibration $\mathfrak{G}_{\mathrm{cyc}} \to \mathfrak{h}/W$ is

$$U := \{(x,v) \in \mathscr{N} \times V | s(x,v) \neq 0\}.$$

The diagonal $\overline{G}$ action on U is transitive and free. Hence $\pi_1(U) = \mathbb{Z}$. We have that $g \cdot s = \det(g)^{-1}s$ for any $g \in \overline{G}$ and thus every simple local system on U of finite order monodromy is given by the horizontal section s^a for some rational number a. We will denote the $\mathcal{D}_U$-module corresponding to such a local system by $\mathcal{E}_a$.

Let $\mathscr{N}_r \subset \mathscr{N}$ denote the regular nilpotent orbit. Then $\pi_1(\mathscr{N}_r) = \mathbb{Z}/n\mathbb{Z}$ and every simple local system on $\mathscr{N}_r$ of finite order monodromy is defined by the representation of $\mathbb{Z}/n\mathbb{Z}$ defined by $e^{2\pi i b}$

for some $b \in \frac{1}{n}\mathbb{Z}$. We will denote the $\mathcal{D}_{\mathscr{N}_r}$-module corresponding to such a local system by $\mathcal{F}_b$. Both $\mathcal{E}_a$ and $\mathcal{F}_b$ are G-equivariant.

The projection $U \to \mathscr{N}$ has image inside $\mathscr{N}_r$ and the fibers of this projection are isomorphic to $\mathfrak{F} := \mathbb{C}^{n-1} \times \mathbb{C}^*$. The fibration

$$\begin{tikzcd} \mathfrak{F} \arrow[r] & U \arrow[d] \\ & \mathscr{N}_r \end{tikzcd}$$

induces an exact sequence

$$1 \to (\mathbb{Z} = \pi_1(\mathfrak{F})) \xrightarrow{n} (\mathbb{Z} = \pi_1(U)) \to (\pi_1(\mathscr{N}_r) = \mathbb{Z}/n\mathbb{Z}) \to 1.$$

It follows that $\mathcal{E}_a$ is the pullback of the local system $\mathcal{F}_a$ on $\mathscr{N}_r$ for $a \in \frac{\mathbb{Z}}{n}$.

2.3.3. *Cuspidal character/mirabolic $\mathcal{D}$-modules.* Let $c = \frac{m}{n}$ for positive integer m coprime to n.

Definition 2.6.
- *The **cuspidal** character $\mathcal{D}_{\mathfrak{g}}$-module of parameter c is the minimal extension of $\mathcal{F}_c$ to $\mathfrak{g}$, which we denote by $\mathbf{N}_c$.*
- *The **cuspidal** mirabolic $\mathcal{D}_{\mathfrak{G}}$-module of parameter c is the minimal extension of $\mathcal{E}_c$ to $\mathfrak{G}$, which we denote by $\overline{\mathbf{N}}_c$.*

By [Lus86], $\mathcal{F}_c$ is clean. We also have the following:

Lemma 2.7. *As* SL_n*-equivariant* $\mathcal{D}_{\mathfrak{G}}$*-modules,* $\overline{\mathbf{N}}_c \cong \mathbf{N}_c \boxtimes \mathcal{O}_V$*. Moreover*

$$SS(\overline{\mathbf{N}}_c) = \{(x, y) \in \mathscr{N} \times \mathscr{N} | [x, y] = 0\}_{\mathrm{red}} \times V \tag{8}$$

where the subscript $_{\mathrm{red}}$ *refers to taking the reduced structure.*

Proof. The function s has degree n along the direction of V. Since $c \cdot n = m$, the local system $\mathcal{E}_c$, defined by the horizontal section s^c, has no monodromy along the V direction. Therefore the minimal extension of $\mathcal{E}_c$ to $\mathscr{N}_r \times V$ is $\mathcal{F}_c \boxtimes \mathcal{O}_V$ and the first statement follows from the cleanness of $\mathcal{F}_c$. On the other hand, it is well-known that the singular support of a cuspidal character $\mathcal{D}$-module equals $\{(x, y) \in \mathscr{N} \times \mathscr{N} | [x, y] = 0\}$ and hence the second statement of the lemma follows. □

Theorem 2.8. *(*[CEE09, Theorem 9.19]*)*
- *There is a* A_{-c-1}*-action on* $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^{\mathrm{SL}_n}$*.*
- *Under this action,* $(\Gamma(\mathfrak{g}, \mathbf{N}_c) \otimes \mathrm{Sym}^m V)^{\mathrm{SL}_n}$ *is isomorphic to* $\Omega_c(\mathrm{eL}_c)$*.*

Corollary 2.9. *As* A_{-c-1}*-modules,* $\mathbb{H}_{-c}(\overline{\mathbf{N}}_c) \cong \Omega_c(\mathrm{eL}_c)$

Proof. Follows from Lemma 2.7, Theorem 2.8 and the fact that $\mathbb{C}[V]^{\tau_{-c}(\mathbf{1})} = \mathrm{Sym}^m V$. □

2.4. **Hilbert schemes of points.** Let $\overline{\mathrm{Hilb}}^n(\mathbb{C}^2)$ denote the moduli space of ideals of colength n in $\mathbb{C}[x, y]$. It is a smooth and quasi-projective variety of dimension $2n$. The Hilbert-Chow map $\mathrm{Hilb}^n \to (\mathbb{C}^2)^n/S_n$, defined by sending a colength n ideal I to the subvariety defined by the quotient $\mathbb{C}[x, y]/I$, is a resolution of singularity. We let $\mathrm{Hilb}^n \subset \overline{\mathrm{Hilb}}^n(\mathbb{C}^2)$ denote the preimage of $\{(z_1, \cdots, z_n) \in (\mathbb{C}^2)^n, \sum z_i = (0, 0)\}$ under the Hilbert-Chow map. Then $\overline{\mathrm{Hilb}}^n = \mathrm{Hilb}^n(\mathbb{C}^2) \times \mathbb{C}^2$.

Define

$$\widetilde{\mathrm{Hilb}^n} := \{(X, Y, v) \in \mathfrak{g} \times \mathfrak{g} \times V | [X, Y] = 0, \mathbb{C}[X, Y]v = V\}_{\mathrm{red}}.$$

The diagonal G-action on $\widetilde{\mathrm{Hilb}}$ is free and the resulting GIT quotient is $\widetilde{\mathrm{Hilb}^n} /\!\!/ G = \mathrm{Hilb}^n$([Nak99]). We also define

$$\widetilde{\mathrm{Hilb}^n_0} = (\mathscr{N} \times \mathscr{N} \times V) \cap \widetilde{\mathrm{Hilb}^n}$$

which is an open subvariety of the singular support (8). The GIT quotient $\mathrm{Hilb}^n_0 := \widetilde{\mathrm{Hilb}^n_0} /\!\!/ G$ is the punctual Hibert scheme which is the zero fiber of the Hilbert-Chow map and is irreducible of dimension $n - 1$.

2.5. **The Gordon-Stafford functor.** Consider the order filtration on H_c defined by $\deg y = 1$ and $\deg x = \deg w = 0$. With respect to this filtration, we have that $\mathrm{gr}(\mathrm{A}_c) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$. We are interested in A_c-modules with good filtrations, in the sense that the associated graded is finitely generated as $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^W$-modules. In [GS05], Gordon and Stafford define a functor from the category of A_c-modules equipped with good filtrations to $\mathrm{Coh}^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2))$, motivated by the following diagram.

$$\begin{tikzcd} ? \arrow[r, rightsquigarrow, "\mathrm{gr}"'] & \mathcal{O}_{\mathrm{Hilb}^n} \\ \mathrm{A}_c \arrow[u] \arrow[r, rightsquigarrow, "\mathrm{gr}"'] & \mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]^{S_n} \arrow[u, "\text{Hilbert-Chow}"'] \end{tikzcd}$$

The definition of their functor GS is based on the Proj construction of the Hilbert scheme due to Haiman [Hai98, Proposition 2.6]. Let $R_0 = \mathbb{C}[\mathfrak{h}]^W$ and for each $k \geq 1$ let $R_k = (\mathbb{C}[\mathfrak{h}]^{\mathrm{sign}})^k$ be the product of k copies of R_1 in $\mathbb{C}[\mathfrak{h}]$. Put $R = \oplus_{k\geq 0} R_k \delta^k$. Then $\mathrm{Hilb}^n = \mathrm{Proj}(R)$. Therefore, any finitely generated graded R-module defines a coherent sheaf on Hilb^n.

Let $\mathrm{e} := \frac{1}{n!}\sum_{w\in W}\mathrm{sign}(w)w$ be the skew-symmetrizing idempotent in H_c. We define two subspaces in $(\mathrm{H}_d)_\delta$:

$${}_{d+1}\mathrm{P}_d := \mathrm{e}\mathrm{H}_d\delta\mathrm{e}_-, \quad {}_d\mathrm{Q}_{d+1} := \mathrm{e}_-\delta^{-1}\mathrm{H}_{d+1}\mathrm{e}.$$

Both ${}_{d+1}\mathrm{P}_d$ and ${}_d\mathrm{Q}_{d+1}$ inherit from $(\mathrm{H}_d)_\delta$ the order filtration. For any $d \in \mathbb{C}$, the isomorphism

$$\mathrm{e}\mathrm{H}_d\mathrm{e} \cong \mathrm{e}\delta^{-1}\mathrm{H}_{d+1}\delta\mathrm{e}$$

gives a $(\mathrm{A}_{d+1}, \mathrm{A}_d)$-module structure on ${}_{d+1}\mathrm{P}_d$ and a $(\mathrm{A}_d, \mathrm{A}_{d+1})$-module structure on ${}_d\mathrm{Q}_{d+1}$. Thus we can inductively define for any $k \in \mathbb{Z}_{>0}$,

$${}_{d+k}\mathrm{P}_d := {}_{d+k}\mathrm{P}_{d+k-1} \otimes_{\mathrm{A}_{d+k-1}} {}_{d+k-1}\mathrm{P}_d,$$

and similarly for ${}_{d-k}\mathrm{Q}_d$. We equip ${}_{d+k}\mathrm{P}_d$ and ${}_{d-k}\mathrm{Q}_d$ with the tensor product filtrations.

For any A_{-d-1}-module L, one has ([GGS09, Proposition 5.8]):

$$(9) \qquad {}_{d+k}\mathrm{P}_d \otimes_{\mathrm{A}_d} \Omega_{-d-1}L \cong \Omega_{-(d+k)-1}({}_{-(d+k)-1}\mathrm{Q}_{-d-1} \otimes_{\mathrm{A}_{-d-1}} L).$$

Moreover, ${}_{c+k}\mathrm{P}_c$ is a $(\mathrm{A}_{c+k}, \mathrm{A}_c)$-module and hence defines a shift functor

$$S_{c,k} : A_c\text{−mod} \to A_{c+k}\text{−mod}, \quad M \mapsto {}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} M.$$

If M is equipped with a filtration, one equips $S_{c,k}(M)$ with the tensor product filtration.

Now for an A_c-module M equipped with a good filtration F, the Gordon-Stafford functor associates to $(M, F_\bullet)$ a coherent sheaf on Hilb^n defined by

$$GS(M,F) = \mathrm{Proj}\left(\mathrm{gr}\bigoplus_{k\geq 0} S_{c,k}M\right)$$

where we take associated graded with respect to the tensor product filtration.

2.6. **Relating the functors.** The parameter c in this subsection can be any complex number.

2.6.1. *Compatibility of filtrations.* Let $\mathbf{M} \in \mathscr{C}(\mathfrak{G})$ be a mirabolic $\mathcal{D}$-module with a good filtration F. Write $(M, F_\bullet) = \Gamma(\mathfrak{G}, (\mathbf{M}, F_\bullet))$. The filtration F restricts to a filtration on $M^{\tau_{-c-k}(\bar{\mathfrak{g}})}$ for any $k \in \mathbb{Z}_{\geq 0}$ and hence also on $L := \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$ as Ω_{-c-1} preserves the order filtration.

For any G-module E, write $E^{\det^{-k}} := \{f \in E | g\cdot f = \det(g)^{-k}f, \forall g \in G\}$. Also, define $\mathcal{D}_d(\mathfrak{G}) := \mathcal{D}(\mathfrak{G})/\mathcal{D}_d(\mathfrak{G})\tau_d(\bar{\mathfrak{g}})$. Then there is a homomorphism

$$(10) \qquad \phi_M^k : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})} \to M^{\tau_{-c-k}(\bar{\mathfrak{g}})}$$

given by the left multiplication of $\mathcal{D}(\mathfrak{G})$ on M. By [BG15, Theorem 1.3.5], ϕ_M^k is an isomorphism of A_{-c-k}-modules.

Moreover, by [GGS09, Theorem 5.3(1)], when each of the rational numbers $-c-1, \cdots, -c-k-1$ satisfies the condition (3), there is an isomorphism

$$\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \cong {}_{-c-k-1}Q_{-c-1} \tag{11}$$

of filtered modules with respect to the order filtrations on $\mathcal{D}(\mathfrak{G})$ and $(\mathrm{H}_c)_\delta$.

Using (9), we have

$$\Omega_{c+k}({}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} L) = {}_{-(c+k)-1}\mathrm{Q}_{-c-1} \otimes_{\mathrm{A}_{-c-1}} \Omega_c(L).$$

The tensor product filtration on the right hand side is induced from the filtration on L and the order filtrations on A_{-c-1} and ${}_{-(c+k)-1}\mathrm{Q}_{-c-1}$. By (11), we have a filtered isomorphism

$$\Omega_{c+k}({}_{c+k}\mathrm{P}_c \otimes_{\mathrm{A}_c} L) \cong \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})}. \tag{12}$$

Consider the tensor product filtration F^T on $\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\mathfrak{g})}$ and the sub-filtration on $M^{\tau_{-c-k}(\mathfrak{g})}$. The homomorphism ϕ^k_M is filtered, i.e., for any $i \geq 0$,

$$\phi^k_M F^T_i \left(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\bar{\mathfrak{g}})}\right) \subset F_i(M^{\tau_{-c-k}(\bar{\mathfrak{g}})}).$$

This is not an equality in general, i.e., ϕ^k_M is not necessarily a filtered isomorphism. Given (12), we see that ϕ^k_M is a filtered isomorphism if and only if

$$\mathrm{gr}^{F^T} S_{c,k} L \cong \mathrm{gr}^F \Omega_{-c-k-1} M^{\tau_{-c-k}(\bar{\mathfrak{g}})}. \tag{13}$$

2.6.2. *A descent functor.* Let $F\mathscr{C}(\mathfrak{G})$ be the category whose objects are pairs $(\mathbf{M}, F_\bullet)$, where $\mathbf{M}$ is a mirabolic $\mathcal{D}$-module and F is a good filtration on $\mathbf{M}$. We define a descent functor

$$\begin{aligned} \Psi_c : F\mathscr{C}^G(\mathfrak{G}) \to& \mathrm{Coh}(\mathrm{Hilb}^n) \\ (\mathbf{M}, F_\bullet) \mapsto& \mathrm{desc}_c(\widetilde{\mathrm{gr}}^F \mathbf{M}|_{\widetilde{\mathrm{Hilb}}^n}) := \mathrm{Proj} \bigoplus_{\ell \geq 0} \Gamma(\widetilde{\mathrm{Hilb}}^n, \mathrm{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\bar{\mathfrak{g}})} \end{aligned} \tag{14}$$

whose essential image lands in $\mathrm{Coh}(\mathrm{Hilb}^n_1)$ where $\mathrm{Hilb}^n_1 \subset \mathrm{Hilb}^n$ is the preimage of $\{(x,0) \in (\mathbb{C}^n)^2\}/S_n$ under the Hilbert-Chow map. Later we will write $\mathrm{desc} := \mathrm{desc}_0$.

2.6.3. *Gradings.* Let $((x_{ij})_{1\leq i,j\leq n}$ and $(\partial_{x_{ij}})_{1\leq i,j\leq n}$ be the dual bases of $\bar{\mathfrak{g}}^*$ and $\bar{\mathfrak{g}}$ respectively. Similarly, let $\{x_i\}$ and $\{y_i\}$ be dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively.

Define $h := \frac{1}{2}\sum_{1\leq i,j\leq n}(x_{ij}\partial_{x_{ij}} + x_{ij}\partial_{x_{ij}})$ and $\mathrm{h}_c := \Omega_{-c-1}(\mathscr{H}_{-c}(h)) = \frac{1}{2}\sum_{i=1}^n (x_i y_i + y_i x_i)$. (The homomorphism $\mathscr{H}_{-c}$ is defined in (7).)

Let $F\mathcal{O}(\mathrm{A}_c)$ be the category of A_c-modules equipped with a good filtration whose essential image under the forgetful functor lies in $\mathcal{O}(\mathrm{A}_c)$. For any $\mathbf{M} \in F\mathscr{C}^G(\mathfrak{G})$, resp. $L \in F\mathcal{O}(\mathrm{A}_c)$, the action of h, resp. h_c, is semisimple and induces a $\mathbb{Z}$-grading on $\mathbf{M}$, resp. L. This grading together with the filtration induces a $\mathbb{C}^* \times \mathbb{C}^*$-equivariant structure on $\Psi_c(\mathbf{M})$, resp. $GS(L)$.

On the other hand, the scalar action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^2$ induces an $\mathbb{C}^* \times \mathbb{C}^*$-action on Hilb^n. We have the following correspondence:

$$\begin{aligned} \text{filtration grading} &\leftrightarrow \left[C^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* :, z \mapsto (1, z)\right] \\ h \text{ or } \mathrm{h}_c\text{-grading} &\leftrightarrow \left[C^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* : z \mapsto (z, z^{-1})\right]. \end{aligned}$$

2.6.4. *Commutativity of the diagram.* Consider the following diagram:

$$\begin{tikzcd} & F\mathscr{C}(\mathfrak{G}) \arrow[dl, "\Omega_{-c-1}\circ\mathbb{H}_{-c}"'] \arrow[dr, "\Psi_c"] & \\ F\mathcal{O}(\mathrm{A}_c) \arrow[rr, "GS"] & & \mathrm{Coh}^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n) \end{tikzcd} \tag{15}$$

Proposition 2.10. *Let $(\mathbf{M}, F_\bullet) \in F\mathscr{C}^G(\mathfrak{G})$ and $(M, F_\bullet) = \Gamma(\mathfrak{G}, (\mathbf{M}, F_\bullet))$. Suppose that $-c-k$ satisfies (3) for all $k \in \mathbb{N}$. Then the following are equivalent:*

(a) $\Psi_c(\mathbf{M}) \cong GS \circ \Omega_{-c-1} \circ \mathbb{H}_{-c}(\mathbf{M})$.
(b) ϕ_M^ℓ *(defined in (10)) is a filtered isomorphism for all* $\ell \gg 0$.
(c) ϕ_M^ℓ *is a filtered isomorphism for all* $\ell \geq 0$.

Proof. Let $L = \Omega_{-c-1} \circ \mathbb{H}_c(M)$. By definition

$$GS(L) = \mathrm{Proj}(\bigoplus_{\ell \geq 0} \mathrm{gr}^{F^T} S_{c,\ell} L)$$

where F^T denotes the tensor product filtration induced by the filtration on L.

On the other hand,

$$\Psi_c(\mathbf{M}) = \mathrm{Proj}(\bigoplus_{\ell \geq 0} \Gamma(\widetilde{\mathrm{Hilb}}, \mathrm{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\overline{\mathfrak{g}})})$$

Therefore, the equality $GS(L) = \Psi_c(\mathbf{M})$ is equivalent to

$$\Gamma(\widetilde{\mathrm{Hilb}}, \mathrm{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\overline{\mathfrak{g}})} \cong \mathrm{gr}^{F^T} S_{c,\ell} L$$

when $\ell \gg 0$.

By [GGS09, Proposition 7.4]

$$\Gamma(\widetilde{\mathrm{Hilb}}, \mathrm{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\overline{\mathfrak{g}})} = \Gamma(T^*\mathfrak{G}, \mathrm{gr}^F \mathbf{M})^{\tau_{-c-\ell}(\overline{\mathfrak{g}})} = \mathrm{gr}^F \Gamma(\mathfrak{G}, \mathbf{M})^{\tau_{-c-\ell}(\overline{\mathfrak{g}})}$$

for $\ell \gg 0$, because $\widetilde{\mathrm{Hilb}}$ is the semistable locus with respect to det.

Since $\Omega_{-c-\ell-1}$ preserves filtration, i.e., $\mathrm{gr}^F \Omega_{-c-\ell-1} M^{\tau_{-c-k}(\overline{\mathfrak{g}})} \cong \mathrm{gr}^F M^{\tau_{-c-\ell}(\overline{\mathfrak{g}})}$, given (13) we conclude that (a) is equivalent to (b).

As for the implication (b) $\Rightarrow$ (c), for any $k \geq 0$, take $\ell \gg 0$ and consider

$$\begin{tikzcd}
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\overline{\mathfrak{g}})} \arrow[r, "id \otimes \phi_M^k"] \arrow[d, "\mathrm{mul} \otimes id"] & \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} M^{\tau_{-c-k}(\overline{\mathfrak{g}})} \arrow[d, "\phi_M^\ell"] \\
\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}} \otimes_{\mathrm{A}_{-c-1}} M^{\tau_{-c}(\overline{\mathfrak{g}})} \arrow[r, "\phi_M^{\ell+k}"] & M^{\tau_{-c-\ell-k}(\overline{\mathfrak{g}})}
\end{tikzcd}$$

Here the map $\mathrm{mul} : \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell}} \otimes_{\mathrm{A}_{-c-k-1}} \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \to \mathcal{D}_{-c}(\mathfrak{G})^{\det^{-\ell-k}}$ is defined by multiplication. By [GGS09, Lemma 5.2(2)], mul is a filtered isomorphism. Also, by assumption, $\phi_M^{\ell+k}$, ϕ_M^ℓ are filtered isomorphisms. As a result, ϕ_M^k is also a filtered isomorphism. This concludes the proof of the proposition. $\square$

3. Hodge filtrations on cuspidal $\mathcal{D}$-modules

From now on, $c = \frac{m}{n}$ for a positive integer m coprime to n.

The structure of a Hodge module [Sai90] is in the form of $(M, F_\bullet, V_\mathbb{Q})$ where M is a D-module, $F_\bullet$ is its Hodge filtration and $V_\mathbb{Q}$ is a $\mathbb{Q}$-perverse sheaf such that $DR(M) = V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$. For an overview, we refer the readers to [Sch]. In this section, we will only adapt a naive notion of $\mathbb{C}$-Hodge modules, in the form of $(M, F_\bullet)$ by forgetting the data of $V_\mathbb{Q}$ in a Hodge module, as in [Sai22, Remark 2.6].

Below, we will have various filtrations but all will be denoted by F if there is no ambiguity.

3.1. **Hodge filtrations.** Fix the Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$ of upper triangular matrices. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent radical and $\mathfrak{n}_r = \mathfrak{n} \cap \mathscr{N}_r$. We give $\mathfrak{n} \times V$ coordinates by

$$\begin{pmatrix} 0 & x_1 & * & * & * & * \\ 0 & 0 & x_2 & * & * & * \\ & & & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{pmatrix} \tag{16}$$

Then the restriction of the function s to $\mathfrak{n} \times V$ equals $x_1 x_2^2 \cdots x_{n-1}^{n-1} v_n^n$ and $\mathfrak{n}_r = \mathfrak{n} \setminus D$ where D is a simple normal crossing (SNC) divisor defined by $D := \{x_1 x_2 \cdots x_{n-1} = 0\}$.

The restriction of $\mathcal{E}_c$ to $U_0 := U \cap (\mathfrak{n} \times V)$ is the local system generated by the horizontal section

$$s^c|_{U_0} = x_c x_2^{2c} \cdots x_{n-1}^{(n-1)c} v_n^m. \tag{17}$$

Similarly, $\mathbf{L}_0 := \mathcal{F}_c|_{\mathfrak{n}_r}$ is defined by the horizontal section

$$s_0^c := x_1^c x_2^{2c} \cdots x_{n-1}^{(n-1)c}. \tag{18}$$

For the order filtration F on $\mathbf{L}_0$, $(\mathbf{L}_0, F)$ defines a variation of Hodge structure.

Write $i_0 : \mathfrak{n}_r \hookrightarrow \mathfrak{n}$. Saito's theory [Sai90] ([Pop18, Theorem 4.3.5]) implies that there exists a unique Hodge module structure on $\mathbf{L}_D := (i_0)_\dagger \mathbf{L}_0$. Following [Pop18, 4.4], Hodge filtrations across SNC divisors can be described explicitly as follows.

First of all, we have that

$$\Gamma(\mathfrak{n}, \mathbf{L}_D) = \mathcal{D}(\mathfrak{n}) / \big(\sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{n})(x_i \partial_{x_i} + ic) + \mathcal{D}(\mathfrak{n}) S([\mathfrak{n}, \mathfrak{n}]) \big).$$

The $\mathcal{D}_{\mathfrak{n}}$-module $\mathbf{L}_D$ is a regular meromorphic extension of $\mathbf{L}_0$ across the SNC divisor D. Inside $\mathbf{L}_D$, we have Deligne's canonical extension $\mathbf{L}_0^{>-1}$ [Del70], which is a locally free $\mathcal{O}_{\mathfrak{n}}$-module extending $\mathbf{L}_0$ such that the residues [HTT08, 5.2.2] of the meromorphic connection under this lattice along all the components of D lie in $(-1, 0]$ and satisfies

$$\mathbf{L}_D = \mathbf{L}_0^{>-1} \otimes_{\mathcal{O}_{\mathfrak{n}}} \mathcal{O}_{\mathfrak{n}}[D] = \mathcal{D}_{\mathfrak{n}} \cdot \mathbf{L}_0^{>-1}$$

Here $\mathcal{O}_{\mathfrak{n}}[D]$ is the sheaf of rational functions on $\mathfrak{n}$ that are regular on U_0.

In our case, for

$$\lceil s_0^c \rceil := x_1^{\lceil c \rceil} x_2^{\lceil 2c \rceil} \dots x_{n-1}^{\lceil (n-1)c \rceil}, \tag{19}$$

we can compute that

$$s_0^c \lceil s_0^c \rceil^{-1} = x_1^{c - \lceil c \rceil} x_2^{2c - \lceil 2c \rceil} \dots x_{n-1}^{(n-1)c - \lceil (n-1)c \rceil}$$

and

$$x_i \partial_{x_i} (s_0^c \lceil s_0^c \rceil^{-1}) = ic - \lceil ic \rceil \in (-1, 0], \quad i = 1, \dots, n-1.$$

Hence $\mathbf{L}_0^{>-1} = \mathcal{O}_{\mathfrak{n}} \lceil s_0^c \rceil^{-1} s_0^c \subset \mathbf{L}_0$.

On $\mathbf{L}_0^{>-1}$ we have the filtration

$$F_k \mathbf{L}_0^{>-1} = \mathbf{L}_0^{>-1} \cap (i_0)_* F_k \mathbf{L}_0 = \mathcal{O}_{\mathfrak{n}} \lceil s_0^c \rceil^{-1}$$

for all $k \geq 0$ and 0 otherwise. Hence the induced filtration on $\mathcal{D}_{\mathfrak{n}} \cdot \mathbf{L}_0^{>-1}$ is

$$F_k(\mathcal{D}_{\mathfrak{n}} \cdot \mathbf{L}_0^{>-1}) = \sum F_i^{\mathrm{ord}} \mathcal{D}_{\mathfrak{n}} F_{k-i} \mathbf{L}_0^{>-1} = (F_k^{\mathrm{ord}} \mathcal{D}_{\mathfrak{n}}) \cdot \lceil s_0^c \rceil^{-1}$$

for all $k \geq 0$ and 0 otherwise.

Let $\mathcal{D}_{\mathfrak{g} \leftarrow \mathfrak{n}} := i^* \mathcal{D}_{\mathfrak{n}} \otimes_{\mathcal{O}_{\mathfrak{n}}} \omega_{\mathfrak{n}/\mathfrak{g}}$. The pushforward of the Hodge module $(\mathbf{L}_D, F_\bullet)$ along $i_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{g}$ has the underlying $\mathcal{D}_{\mathfrak{g}}$-module

$$\mathbf{L} = (i_{\mathfrak{n}})_\dagger \mathbf{L}_D = i_* \big(\mathcal{D}_{\mathfrak{g} \leftarrow \mathfrak{n}} \otimes_{\mathcal{D}_{\mathfrak{n}}} \mathbf{L}_D \big)$$

with the Hodge filtration filtration on $\mathbf{L}$ defined by the formula ([Pop18, 1.5])

$$F_k(i_{\mathfrak{n}})_\dagger(\mathbf{L}_D) = \mathrm{Im}\Big(\Big(\sum_q F_q^{\mathrm{ord}}\mathcal{D}_{\mathfrak{g}\leftarrow\mathfrak{n}} \otimes F_{k-q}^D(\mathbf{L}_D)\Big) \to (i_{\mathfrak{n}})_\dagger(\mathbf{L}_D)\Big).$$

Let $\overline{\mathbf{L}}$ be the minimal extension of $\mathcal{E}_c|_{U_0}$ to $\mathfrak{G}$. One can run the same procedure to define a $\mathbb{C}$-Hodge module structure on $\overline{\mathbf{L}}$.

3.2. **Associated graded of L.** Recall the standard $\mathfrak{sl}_2$-triple of SL_n:

$$\underline{e} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \underline{f} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ n-1 & 0 & \cdots & 0 & 0 \\ 0 & 2(n-2) & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1-n & 0 \end{pmatrix} \quad \underline{h} = [E, F] \tag{20}$$

Define the Hessenberg space $\mathfrak{m} = \overline{B \cdot (\underline{f} + \mathrm{stab}_{\mathfrak{g}}(\underline{e}))}$, which is the sum of $\mathfrak{m}$ and the span of negative simple root spaces.

Remark 3.1. *The affine space $F + \mathrm{stab}_{\mathfrak{g}}(E)$ is the Kostant slice. The Whittaker reduction of $G \times^B \mathfrak{m}$ is the compactified regular centralizer* [Ba23].

Under the coordinates (16) and

$$\begin{pmatrix} * & * & \cdots & * & * & * \\ y_1 & * & \cdots & * & * & * \\ 0 & y_2 & \cdots & * & * & * \\ & & \cdots & & & \\ 0 & 0 & \cdots & y_{n-2} & * & * \\ 0 & 0 & \cdots & 0 & y_{n-1} & * \end{pmatrix} \tag{21}$$

we define

$$\mathfrak{Y}_0 := \{(x, y) \in \mathfrak{n} \times \mathfrak{m} | x \in \mathfrak{n}, y \in \mathfrak{m}, x_i y_i = 0, 1 \le i \le n-1\}.$$

As a result of the identification:

$$\Gamma(\mathfrak{g}, \mathbf{L}) \cong \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot \mathcal{O}(\mathfrak{b}_-) + \sum_{i=1}^{n-1} \mathcal{D}(\mathfrak{g})(x_i\partial_{x_i} - ic) + \mathcal{D}(\mathfrak{g}) \cdot S([\mathfrak{n}, \mathfrak{n}]),$$

we have that $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{L} \cong (i_{\mathfrak{Y}_0})_* \mathcal{O}_{\mathfrak{Y}_0}$, where $i_{\mathfrak{Y}_0} : \mathfrak{Y}_0 \to T^*\mathfrak{g}$.

Although $\mathbf{L}$ and $\overline{\mathbf{L}}$ are only B-equivariant but not $\overline{B}$-equivariant, the coherent sheaf $\widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$ are naturally $\overline{B}$-equivariant when we pre-compose the original $\bar{\mathfrak{b}}$-action with τ_c so that it is integrable. This definition of $\overline{B}$-equivariance is consistent with the definition of Ψ_c (14). Below, we will further describe such equivariant structure on $\widetilde{\mathrm{gr}}\ \mathbf{L}$, resp. $\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$.

Let α be the weight associated to the relative canonical bundle $\omega_{\mathfrak{n}/\mathfrak{g}}$ and $\gamma_n = (0, 0, \cdots, 0, 1)$. Let $i_V : V \hookrightarrow T^*V$ be the zero section. Endow $\mathcal{O}_{\mathfrak{Y}_0}$, resp. $\mathcal{O}_{\mathfrak{Y}_0} \boxtimes (i_V)_*\mathcal{O}_V$ with the trivial B, resp. $\overline{B}$-equivariant structure. Let $\mathbb{C}_\lambda$ be the 1-dimensional representation of $\overline{B}$ associated to the character λ. Denote the embedding $\mathfrak{Y}_0 \to \mathfrak{g} \times \mathfrak{g}$ by $i_{\mathfrak{Y}_0}$.

Moreover, we write down the important weights:

$$\lceil \mu_c \rceil = (\lceil \mu_c \rceil(1), \cdots, \lceil \mu_c \rceil(n)) := (\lceil c \rceil, \lceil 2c \rceil - \lceil c \rceil, \ldots, \lceil nc \rceil - \lceil (n-1)c \rceil). \tag{22}$$

and

$$\lfloor \mu_c \rfloor = (\lfloor \mu_c \rfloor(1), \cdots, \lfloor \mu_c \rfloor(n)) := (\lfloor c \rfloor, \lfloor 2c \rfloor - \lfloor c \rfloor, \ldots, \lfloor nc \rfloor - \lfloor (n-1)c \rfloor) \tag{23}$$

satisfying $\lfloor \mu_c \rfloor = w_0 \lceil \mu_c \rceil$ where w_0 is the longest element in W.

Lemma 3.2. *As $\overline{B}$-equivariant $\mathcal{O}_{T^*\mathfrak{g}}$-modules, $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{L} = (i_{\mathfrak{Y}_0})_*\mathcal{O}_{\mathfrak{Y}_0} \otimes \mathbb{C}_{\lceil \mu_c \rceil + \alpha - m\gamma_0}$.*

Proof. First of all, we have mentioned above that $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{L} \cong (i_{\mathfrak{Y}_0})_*\mathcal{O}_{\mathfrak{Y}_0}$. Moreover, note that the first non-vanishing filtered piece of $\mathrm{gr}^{\mathrm{H}}\mathbf{L}$ equals $\omega_{\mathfrak{n}/\mathfrak{g}} \otimes \mathbb{C}\lceil s_0^c \rceil^{-1}$. The lemma follows from that $\overline{B}$-acts on $\lceil s_0^c \rceil^{-1}$ exactly by the character $\lceil \mu_c \rceil - m\omega_0$. ☐

We similarly have that:

Lemma 3.3. *As $\overline{B}$-equivariant $\mathcal{O}_{T^*\mathfrak{G}}$-modules,* $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{L}} = \big((i_{\mathfrak{Y}_0})_*\mathcal{O}_{\mathfrak{Y}_0} \boxtimes (i_V)_*\mathcal{O}_V\big) \otimes \mathbb{C}_{\lceil \mu_c \rceil + \alpha}$.

Proof. Comparing Lemmas 3.2 and 3.3, the extra factor of $m\gamma_n$ comes from the difference between $s^c|_{U_0}$ (eq. (17)) and s_0^c (eq. (18)). ☐

3.3. **Functors on equivariant Hodge modules.** In this section, we state two important results about Hodge modules, which allow us to define the Hodge filtration and describe the associated graded of the cuspidal D-modules later.

Let G be an affine algebraic group and X be a smooth variety with a G-action. We denote the category of G-equivariant Hodge modules of weight ℓ on a smooth variety X by $\mathrm{HM}^G(X,\ell)$ and refer the readers to [Ach, Chapter 5] for a definition.

3.3.1. *Inductions.* Let H be a closed subgroup of G, which acts on the product $G \times X$ by $h\cdot(g,x) = (gh^{-1}, hx)$. Let $G \times^H X$ be the quotient of this action. Consider the diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\pi"] \arrow[d, "pr"] & G \times^H X \arrow[d, "a"] \\ X & X \end{tikzcd}$$

where pr is the second projection, π is the quotient map and $a : \overline{(g,x)} \mapsto gx$.

For any H-equivariant $\mathcal{D}_X$-module $\mathcal{F}$, there is a unique G-equivariant $\mathcal{D}_{G\times^H X}$-module $\mathcal{E}$ such that $pr^\dagger\mathcal{F} \cong \pi^\dagger\mathcal{E}$ [BL94]. We denote $\mathcal{E} = \mathrm{Ind}_H^G\mathcal{F}$ and $\widetilde{\mathrm{Ind}}_H^G := a_\dagger \circ \mathrm{Ind}_H^G$.

Similarly, for any H-equivariant coherent $\mathcal{O}_X$-module $\mathcal{F}$, we write $\mathrm{ind}_H^G\mathcal{F}$ to denote the corresponding G-equivariant $\mathcal{O}_{G\times^H X}$-module.

For any map $f : X \to Y$, we introduce the associated Lagrangian correspondence as indicated by (24): the map ρ_f is the co-differential of f and ϖ_f is the natural projection.

$$T^*X \xleftarrow{\rho_f} X \times_Y T^*Y \xrightarrow{\varpi_f} T^*Y. \tag{24}$$

Combining the cases when $f = \pi$ or $f = pr$, we obtain a commutative diagram

$$\begin{tikzcd} & T^*(G \times X) & G \times T^*X \arrow[l, hook', "\rho_{pr}"'] \arrow[r, two heads, "\varpi_{pr}"] & T^*X \\ T^*(G\times^H X)\times_{G\times^H X}(G\times X) \arrow[ur, hook, "\rho_\pi"] \arrow[rr, two heads, "\varpi_\pi"] & & T^*(G\times^H X) & G\times^H T^*X \arrow[l, hook', "s"'] \end{tikzcd} \tag{25}$$

Here s is the zero section.

Proposition 3.4. *For any $\mathbf{M} \in \mathrm{HM}^G(G\times^H X)$, $SS(\mathbf{M}) \subset s(G\times^H T^*X)$ and the following diagram commutes, where $\widetilde{\mathrm{gr}}$ is taken with respect to Hodge filtration.*

$$\begin{tikzcd} \mathrm{HM}^H(X,\ell) \arrow[r, "\mathrm{Ind}_H^G", "\sim"'] \arrow[d, "\widetilde{\mathrm{gr}}"] & \mathrm{HM}^G(G\times^H X,\ell) \arrow[d, "\widetilde{\mathrm{gr}}"] \\ \mathrm{Coh}^H(T^*X) \arrow[r, "\mathrm{ind}_H^G", "\sim"'] & \mathrm{Coh}^G(G\times^H T^*X) \end{tikzcd}$$

Proof. That the functor Ind_H^G is an isomorphism in the level of Hodge modules can be found in [Ach, Theorem 6.2]. It remains to show the essential image of $\mathrm{HM}^G(G\times^H X)$ under $\widetilde{\mathrm{gr}}$ is as desired and the diagram is commutative.

Suppose $\mathbf{M} = \mathrm{Ind}_H^G\mathbf{L}$ for $\mathbf{L}\in \mathrm{HM}^H(X)$. By definition, $\pi^\dagger\mathbf{M} = pr^\dagger\mathbf{L}$ as G-equivariant $\mathcal{D}_{G\times X}$-modules. Taking associated graded of both sides and using [Kas03, Theorem 4.7], we obtain an isomorphism in $\mathrm{Coh}^G(T^*(G\times X))$

$$(\rho_\pi)_*\varpi_\pi^*\widetilde{\mathrm{gr}}\ \mathbf{M} \cong (\rho_{pr})_*\varpi_{pr}^*\widetilde{\mathrm{gr}}\ \mathbf{L}. \tag{26}$$

In particular, we see $SS(\mathbf{M})\subset \varpi_\pi\rho_\pi^{-1}(G\times T^*X)\subset s(G\times^H T^*X)$.

Given the diagram (25), the identity (26) implies $\varpi_{pr}^*\widetilde{\mathrm{gr}}\ \mathbf{L} = q^*\widetilde{\mathrm{gr}}\ \mathbf{M}$. Therefore $\mathrm{ind}_B^G\widetilde{\mathrm{gr}}\ \mathbf{L} = \widetilde{\mathrm{gr}}\ \mathbf{M}$ and the diagram in the statement commutes. □

3.3.2. *Associated graded of a pushforward.* Let $p: X\to Y$ be a projective morphism between two smooth varieties and $M\in \mathrm{HM}(X,\ell)$. By the directed image theorem of Saito ([Sai88]), one has that $R^ip_\dagger(M)\in \mathrm{HM}(Y,\ell+i)$. Let $\widetilde{\omega_{X/Y}}$ denote the pullback of the relative canonical bundle $\omega_{X/Y}$ to $X\times_Y T^*Y$.

Theorem 3.5. *([Lau83, 2.3.2], [Sch, 28]) The diagram below commutes*

$$\begin{tikzcd} \mathrm{HM}^G(X,\ell) \arrow[r, "R^ip_\dagger"] \arrow[d, "\widetilde{\mathrm{gr}}"] & \mathrm{HM}^G(Y,\ell+i) \arrow[d, "\widetilde{\mathrm{gr}}"] \\ \mathrm{Coh}^G(T^*X) \arrow[r, "R^i(\varpi_p)_*(\widetilde{\omega_{X/Y}}\otimes L\rho_p^*(-))"] & \mathrm{Coh}^G(T^*Y) \end{tikzcd}$$

where $\widetilde{\mathrm{gr}}$ is taken with respect to the Hodge filtration and ρ_p,ϖ_p are defined by (24).

Remark 3.6. *Note that this result does not hold in general if $\widetilde{\mathrm{gr}}$ is taken with respect to an arbitrary good filtration.*

3.4. **Springer cuspidal D-modules.** We apply the two results in the last section to study the cuspidal D-modules $\mathbf{N}_c$ and $\overline{\mathbf{N}}_c$.

Since every regular nilpotent x is contained in a unique Borel subalgebra, there are embeddings of $\mathscr{N}_r$ into $\mathcal{B}\times\mathfrak{g}$ and of U into $\mathcal{B}\times\mathfrak{G}$:

$$\begin{tikzcd} \mathscr{N}_r \arrow[r, hook] \arrow[dr, hook] & \mathfrak{g} \\ & \mathcal{B}\times\mathfrak{g} \arrow[u, "p"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[dr, hook, "\widetilde{i}"'] & \mathfrak{G} \\ & \mathcal{B}\times\mathfrak{G} \arrow[u] \end{tikzcd}$$

Definition 3.7. *The Springer cuspidal $\mathcal{D}_\mathfrak{g}$-module $\mathbf{M}_c$ is the minimal extension of $\mathcal{F}_c$ to $\mathcal{B}\times\mathfrak{g}$. The Springer cuspidal $\mathcal{D}_\mathfrak{G}$-module $\overline{\mathbf{M}}_c$ is the minimal extension of $\mathcal{E}_c$ to $\mathcal{B}\times\mathfrak{G}$.*

The local system $\mathcal{F}_c$ is also clean with respect to the inclusion $\widetilde{i}$, i.e., $\mathbf{M}_c = \widetilde{i}_\dagger\mathcal{F}$. Therefore by functoriality $p_\dagger\mathbf{M}_c = \mathbf{N}_c$. Similar to Lemma 2.7, we have that $\overline{\mathbf{M}}_c = \mathbf{M}_c\boxtimes\mathcal{O}_V$.

Lemma 3.8. *(1) The G-equivariant $\mathcal{D}$-module underlying $\mathrm{Ind}_B^G(\mathbf{L},F_\bullet)$ is $\mathbf{M}_c$.*
(2) The G-equivariant $\mathcal{D}$-module underlying $\mathrm{Ind}_B^G(\overline{\mathbf{L}},F_\bullet)$ is $\overline{\mathbf{M}}_c$.

Proof. We only show (1) and the same argument applies to (2) since $\overline{\mathbf{L}}\cong\mathbf{L}\boxtimes\mathcal{O}_V$. Consider the following cartesian diagram

$$\begin{tikzcd} \mathfrak{n}_r \arrow[d, "i_{0,r}"] & G\times\mathfrak{n}_r \arrow[l, "p_0"'] \arrow[r, "\pi_0"] \arrow[d, "i_r"] & G\times^B\mathfrak{n}_r \arrow[r, equal] & \mathscr{N}_r \arrow[d, "i"] \\ \mathfrak{g} & G\times\mathfrak{g} \arrow[l, "p"'] \arrow[rr, "\pi"] & & \mathcal{B}\times\mathfrak{g} \end{tikzcd}$$

Since $\mathcal{E}_c = \mathrm{Ind}_B^G \mathbf{L}_0$, using base change twice, we have

$$\pi^\dagger i_\dagger(\mathcal{E}_c) = (i_r)_\dagger \pi_0^\dagger(\mathcal{E}_c) = (i_r)_\dagger p_0^\dagger(\mathcal{F}_c) = p^\dagger(i_{0,r})_\dagger(\mathcal{F}_c)$$

which proves the lemma. $\square$

Consider $\mathfrak{Y} = G \times^B \mathfrak{Y}_0$ with embedding $i_{\mathfrak{Y}} : \mathfrak{Y} \to T^*(\mathcal{B} \times \mathfrak{g})$ such that the following diagram is Cartesian:

$$\begin{tikzcd} \mathfrak{Y}_0 \arrow[r] \arrow[d] & \mathfrak{g} \times \mathfrak{n} \times \mathfrak{g} \arrow[d] \\ G \times^B \mathfrak{Y}_0 \arrow[r, "i_{\mathfrak{Y}}"] & G \times^B (\mathfrak{g} \times \mathfrak{n} \times \mathfrak{g}) \end{tikzcd}$$

Let $\mathcal{L}_\lambda$ be the $\overline{G}$-equivariant line bundle on $\mathcal{B}$ associated to the weight λ. Write $\widetilde{\mathcal{L}}_\lambda := (\pi_{T^*(\mathcal{B}\times\mathfrak{g})\to\mathcal{B}})^*\mathcal{L}_\lambda$. Endow $(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}$, resp. $(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}} \boxtimes \mathcal{O}_V$, with the trivial G, resp. $\overline{G}$-equivariant structure.

We consider the $\overline{G}$-equivariant structure on $\widetilde{\mathrm{gr}}\ \overline{\mathbf{M}}_c$, resp. $\widetilde{\mathrm{gr}}\ \overline{\mathbf{N}}_c$ later, by precomposing the original $\overline{\mathfrak{g}}$-action with τ_c so that it is integrable.

Corollary 3.9. *We have the following $\overline{G}$-equivariant isomorphisms:*

- $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{M}_c \cong \widetilde{\mathcal{L}}_{\lceil\mu_c\rceil+\alpha-m\gamma_n} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}$;
- $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{M}}_c \cong \big(\widetilde{\mathcal{L}}_{\lceil\mu_c\rceil+\alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\big) \boxtimes (i_V)_*\mathcal{O}_V$.

Proof. By Proposition 3.4 and Lemma 3.8, we have $\widetilde{\mathrm{gr}}\ \mathbf{M}_c \cong \mathrm{ind}_B^G\widetilde{\mathrm{gr}}\ \mathbf{L}$ and $\widetilde{\mathrm{gr}}\ \overline{\mathbf{M}}_c \cong \mathrm{ind}_B^G\widetilde{\mathrm{gr}}\ \overline{\mathbf{L}}$. It remains to use Lemma 3.2 and Lemma 3.3. $\square$

4. Cuspidal DG modules and bigraded characters

4.1. **Cuspidal DG modules.** Recall the coordinates from (16) and (21). When $x \in \mathfrak{n}$ and $y \in \mathfrak{m}$, we have $[x, y] \in \mathfrak{b}$. Moreover, the diagonals of $[x, y]$ equal $x_1y_1,\ x_2y_2 - x_1y_1, \ldots,\ -x_{n-1}y_{n-1}$. As a result, $x_iy_i = 0$, $1 \le i \le n-1$ if and only if $[x, y] = 0 \bmod \mathfrak{n}$. That is to say, the following diagram is Cartesian:

(27)
$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "q_{\mathfrak{n}}"] \arrow[d] & G \times^B \mathfrak{n} \arrow[d] \\ G \times^B (\mathfrak{n} \times \mathfrak{m}) \arrow[r, "q_{\mathfrak{b}}"] & G \times^B \mathfrak{b} \end{tikzcd}$$

with $q_{\mathfrak{b}} : G \times^B (\mathfrak{n} \times \mathfrak{m}) \to G \times^B \mathfrak{b}$ defined by $(g, x, y) \mapsto (g, [x, y])$ and $q_{\mathfrak{n}}$ is the restriction of $q_{\mathfrak{b}}$ to $\mathfrak{Y}$.

On $\mathcal{B}$ we have the vector bundle $\underline{\mathfrak{b}^*}$ (resp. $\underline{\mathfrak{b}}$) whose total space equals $G \times^B \mathfrak{b}^*$ (resp. $G \times^B \mathfrak{b}$). Let $\pi_{\mathfrak{b}} : G \times^B \mathfrak{b} \to \mathcal{B}$ be the projection and $\iota_{\mathfrak{b}} : \mathcal{B} \to G \times^B \mathfrak{b}$ be the zero section. The Koszul complex $(\wedge^\bullet \pi_{\mathfrak{b}}^* \underline{\mathfrak{b}^*}, \partial_{\mathfrak{b}})$, with differential $\partial_{\mathfrak{b}}$ defined by contraction with the canonical section of $\pi_{\mathfrak{b}}^*\underline{\mathfrak{b}}$, is quasi-isomorphic to $(\iota_{\mathfrak{b}})_*\mathcal{O}_{\mathcal{B}}$.

One can similarly define $\pi_{\mathfrak{n}}$, $\iota_{\mathfrak{n}}$, $\underline{\mathfrak{n}^*}$, $\partial_{\mathfrak{n}}$, such that $(\wedge^\bullet \pi_{\mathfrak{n}}^*\underline{\mathfrak{n}^*}, \partial_{\mathfrak{n}})$ is quasi-isomorphic to $(\iota_{\mathfrak{n}})_*\mathcal{O}_{\mathcal{B}}$.

We define a DG algebra by

$$\mathcal{A}'' := ((\wedge^\bullet(\pi_{\mathfrak{n}} \circ q_{\mathfrak{n}})^*\underline{\mathfrak{n}^*}, q_{\mathfrak{n}}^*\partial_{\mathfrak{n}})).$$

By definition, the associated DG scheme $\mathit{Spec}(\mathcal{A}'')$ makes the following diagram Cartesian.

(28)
$$\begin{tikzcd} \mathit{Spec}(\mathcal{A}'') \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "\iota_{\mathfrak{n}}"] \\ \mathfrak{Y} \arrow[r, "q_{\mathfrak{n}}"] & G \times^B \mathfrak{n} \end{tikzcd}$$

Given the Cartesian diagram (27), we have that $(i_{\mathfrak{Y}\to G\times^B(\mathfrak{n}\times\mathfrak{m})})_*\mathcal{A}''$ is quasi-isomorphic to

$$\mathcal{A} := ((\wedge^\bullet q_{\mathfrak{b}}^*\pi_{\mathfrak{b}}^*\underline{\mathfrak{b}^*}, q_{\mathfrak{b}}^*\partial_{\mathfrak{b}})).$$

Because (28) and the diagram on the left of (29) of are Cartesian, diagram on the right of (29) is also Cartesian.

$$\begin{tikzcd} G\times^B T^*\mathfrak{g} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ T^*(\mathcal{B}\times\mathfrak{g}) \arrow[r] & G\times^B\mathfrak{n} \end{tikzcd} \qquad \begin{tikzcd} Spec(\mathcal{A}'') \arrow[r] \arrow[d] & \mathcal{B}\times T^*\mathfrak{g} \arrow[d] \\ \mathfrak{Y} \arrow[r] & T^*(\mathcal{B}\times\mathfrak{g}) \end{tikzcd} \tag{29}$$

Diagrams (27), (28) and (29) can be combined into Figure 1.

$$\begin{tikzcd}
 & & Spec(\mathcal{A}'') \arrow[rr] \arrow[dl] \arrow[ddrr, bend right] & & \mathcal{B} \arrow[dl, "\iota_{\mathfrak{n}}"'] \\
 & \mathfrak{Y} \arrow[rr, "q_{\mathfrak{n}}"] \arrow[dl] \arrow[ddrr, bend right] & & G\times^B \mathfrak{n} \arrow[dl] & \\
G\times^B(\mathfrak{n}\times\mathfrak{m}) \arrow[rr, "q_{\mathfrak{b}}"] & & G\times^B\mathfrak{b} & & \mathcal{B}\times T^*\mathfrak{g} \arrow[uu, "\pi_{T^*\mathfrak{g}}"'] \arrow[dl] \\
 & & & T^*(\mathcal{B}\times\mathfrak{g}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. Cartesian diagrams

Let $\pi_{G\times^B(\mathfrak{n}\times\mathfrak{m})\to\mathcal{B}}$ and $\pi_{\mathfrak{Y}\to\mathcal{B}}$ be projections to $\mathcal{B}$.

Definition 4.1. *The cuspidal DG module of slope c is*

$$\mathcal{A}_c := \mathcal{A}\otimes(\pi_{G\times^B(\mathfrak{n}\times\mathfrak{m})\to\mathcal{B}})^*\mathcal{L}_{\lceil\mu_c\rceil} \tag{30}$$

Similarly, define $\mathcal{A}''_c := \mathcal{A}''\otimes(\pi_{\mathfrak{Y}\to\mathcal{B}})^*\mathcal{L}_{\lceil\mu_c\rceil}$.
Consider the following maps:

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r, "i_{\mathfrak{Y}}"] \arrow[dr, "p_{\mathfrak{Y}}"'] & T^*(\mathcal{B}\times\mathfrak{g}) \arrow[d, "p_{T^*(\mathcal{B}\times\mathfrak{g})}"] & \mathcal{B}\times T^*\mathfrak{g} \arrow[l, "i_{\mathcal{B}\times T^*\mathfrak{g}}"'] \arrow[dl, "p_{\mathcal{B}\times T^*\mathfrak{g}}"] \\
 & T^*\mathfrak{g} &
\end{tikzcd}$$

Here $p_{\mathfrak{Y}}:\mathfrak{Y}\to T^*\mathfrak{g}$ is the restriction of $p: G\times^B(\mathfrak{n}\times\mathfrak{m})\to T^*\mathfrak{g}:(g,x,y)\mapsto(g\cdot x, g\cdot y)$.

Proposition 4.2. *There is a $\overline{G}$-equivariant isomorphism,*

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c = Rp_*\mathcal{A}_c\boxtimes(i_V)_*\mathcal{O}_V.$$

Proof. By Theorem 3.5, we have a G-equivariant isomorphism:

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{N}_c = R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*Li^*_{\mathcal{B}\times T^*\mathfrak{g}}(\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathbf{M}_c).$$

The identification from Corollary 3.9: $\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{M}}_c = (\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor+\alpha}\otimes(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}})\boxtimes(i_V)_*\mathcal{O}_V$ implies:

$$\widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c = R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*\left(Li^*_{\mathcal{B}\times T^*\mathfrak{g}}\big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor+\alpha}\otimes(i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\big)\otimes\pi^*_{T^*\mathfrak{g}}\omega_{\mathcal{B}}\right)\boxtimes(i_V)_*\mathcal{O}_V \tag{31}$$

Let us cite the following result, which is a consequence of base changes:
([Gin12, Lemma 4.4.1]) *Suppose X is a smooth variety and $i_Y : Y \to X$, $i_Z : Z \to X$ are embeddings of closed subvarieties. Then:*

$$(i_Y)_* Li_Y^*(i_Z)_*\mathcal{O}_Z = (i_Y)_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} (i_Z)_*\mathcal{O}_Z = (i_Z)_* Li_Z^*(i_Y)_*\mathcal{O}_Y.$$

Applying this lemma to (31) in the setting of $Y = \mathcal{B} \times T^*\mathfrak{g}$ and $Z = \mathfrak{Y}$, we obtain:

$$\begin{aligned}
&R(p_{\mathcal{B}\times T^*\mathfrak{g}})_*\Big(Li^*_{\mathcal{B}\times T^*\mathfrak{g}}\big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor+\alpha} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}}\big) \otimes \pi^*_{T^*\mathfrak{g}}\omega_{\mathcal{B}}\Big)\\
=&\mathrm{R}(p_{T^*(\mathcal{B}\times\mathfrak{g})})_*\Big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor} \otimes (i_{\mathfrak{Y}})_*\mathcal{O}_{\mathfrak{Y}} \otimes (i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big)\\
=&R(p_{T^*(\mathcal{B}\times\mathfrak{g})})_* R(i_{\mathfrak{Y}})_* L(i_{\mathfrak{Y}})^*\Big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor} \otimes (i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big)\\
=&R(p_{\mathcal{B}\times T^*\mathfrak{g}})_* L(i_{\mathfrak{Y}})^*\Big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor} \otimes (i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big). \qquad (32)
\end{aligned}$$

Since the diagram on the right of (29) is Cartesian, we have

$$L(i_{\mathfrak{Y}})_*\Big(\widetilde{\mathcal{L}}_{\lfloor\mu_c\rfloor} \otimes (i_{\mathcal{B}\times T^*\mathfrak{g}})_*\mathcal{O}_{\mathcal{B}\times T^*\mathfrak{g}}\Big) = R(i_{\mathfrak{Y}})_*\mathcal{A}''_c.$$

Therefore, (32) equals

$$R(p_{\mathcal{B}\times T^*\mathfrak{g}})_* R(i_{\mathfrak{Y}})_*\mathcal{A}''_c = R(p_{\mathfrak{Y}})_*\mathcal{A}''_c = Rp_*\mathcal{A}_c.$$

The proposition follows. □

4.2. **Equivariant K-theory of Hilbert schemes.** Write $A := \mathbb{C}^* \times \mathbb{C}^*$. Let $K^A(\mathrm{Hilb}^n)$ denote the equivariant K-theory group, which is a module over $K^A(pt) = \mathbb{C}[q^\pm, t^\pm]$.

Define the isospectral Hilbert scheme IHilb^n to be the reduced fibered product of the following diagram

$$\begin{array}{ccc}
\mathrm{IHilb}^n & \xrightarrow{\beta} & \mathbb{C}^{2n}\\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow\\
\mathrm{Hilb}^n & \longrightarrow & \mathbb{C}^{2n}/S_n
\end{array}$$

A deep result of Haiman states that $\mathcal{P} := \alpha_*\mathcal{O}_{\mathrm{IHilb}^n}$ is a vector bundle of rank $n!$, which is known since as the Procesi bundle $\mathcal{P}$. As a corollary, one has the following isomorphism between K-groups:

$$\beta_*\alpha^* : K^A(\mathrm{Hilb}^n) \cong K^{S_n\times A}(\mathbb{C}^{2n}). \qquad (33)$$

By base change, $\beta_*\alpha^*(\mathcal{F}) \cong \Gamma(\mathrm{Hilb}^n, \mathcal{F} \otimes \mathcal{P})$.

The Grothendieck group $K^{S_n\times A}(\mathbb{C}^{2n})$ is freely generated by $V_\lambda \otimes \mathbb{C}[\mathbb{C}^{2n}]$ over $\mathbb{C}[q^\pm, t^\pm]$ where V_λ is the irreducible representation of S_n associated to $\lambda \vdash n$. Let s_λ be the Schur function associated to λ. Then the bigraded Frobenius character of $V_\lambda \otimes \mathbb{C}[\mathbb{C}^{2n}]$ is

$$\mathrm{ch}_{S_n\times\mathbb{C}^*\times\mathbb{C}^*}(V_\lambda \otimes \mathbb{C}[\mathbb{C}^{2n}]) = s_\lambda\big(\frac{z}{(1-q)(1-t)}\big).$$

Therefore, composing (33) with $\mathrm{ch}_{S_n\times\mathbb{C}^*\times\mathbb{C}^*}$ establishes an isomorphism

$$\kappa : K^A(\mathrm{Hilb}^n) \cong \{f \in \mathbb{C}(q,t)[z_1, \cdots, z_n]^{S_n} | f((1-q)(1-t)z) \text{ has coefficients in } \mathbb{C}[q^\pm, t^\pm]\} \qquad (34)$$

such that $\kappa(\mathcal{V}_\lambda) = s_\lambda(\frac{z}{(1-q)(1-t)})$ where $\mathcal{V}_\lambda = \mathrm{Hom}_{S_n}(V_\lambda, \mathcal{P})$.

Let λ^t denote the transpose of the partition $\lambda \vdash n$. The modified Macdonald polynomials $\widetilde{H}_\lambda(z;q,t)$ are the unique symmetric polynomials satisfying

$$\widetilde{H}_\lambda((1-q)z;q,t) \in \mathbb{C}(q,t)\{s_\mu | \mu \geq \lambda\};$$
$$\widetilde{H}_\lambda((1-t)z;q,t) \in \mathbb{C}(q,t)\{s_\mu | \mu \geq \lambda^t\};$$
$$(\widetilde{H}_\lambda(z;q,t), s_{(n)}) = 1.$$

The A-fixed points in Hilb^n are in bijection with partitions of n. For any $\lambda \vdash n$, let I_λ be the associated fixed point and $[I_\lambda]$ be the K-theory class corresponding to the skyscraper sheaf supported on I_λ.

Proposition 4.3. *([Hai03, Theorem 4.1.5 and Proposition 5.4.1]) The image of $[I_\lambda]$ under (34) is $\widetilde{H}_\lambda$.*

Throughout, we may not distinguish between partitions and Young diagrams.

For a box x inside a Young diagram σ, let a, ℓ, resp. a', ℓ', denote its arm and leg, resp. coarm and coleg (demonstrated in Figure (2)). For a Young tableau, define the weight of the box x labeled i by $\chi_i = q^{a'(x)}t^{l'(x)}$.

Define

$$g_\lambda = \prod_{x\in\sigma}(1-q^{a(x)+1}t^{-\ell(x)})(1-q^{-a(x)}t^{\ell(x)+1})$$

which is the bigraded character of the cotangent space at I_λ in $\mathrm{Hilb}^n(\mathbb{C}^2)$.

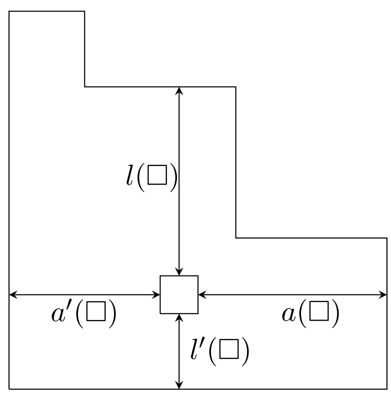


FIGURE 2. ([GN15, Fig.1])Arm, leg, co-arm and co-leg

Let $\iota_\lambda : \{I_\lambda\} \hookrightarrow \mathrm{Hilb}^n$ denote the embedding. For any $[\mathcal{F}] \in K^A(\mathrm{Hilb}^n)$, under the isomorphism (34) we have the localization formula ([CG10, Proposition 5.10.3]):

$$\kappa([\mathcal{F}]) = \sum_{\lambda\vdash n} \frac{\widetilde{H}_\lambda}{g_\lambda} \mathrm{ch}_A(L\iota_\lambda^*\mathcal{F}). \tag{35}$$

4.3. Bigraded character of L_c.

4.3.1. *Principal nilpotent pairs.* In [Gin00], a pair of commuting elements (x_1, x_2) in $\mathfrak{g}\times\mathfrak{g}$ is called a principal nilpotent pair if

- (x_1, x_2) is regular, i.e., the joint centralizer of x_1 and x_2 is of minimal dimension.
- For any $(t_1, t_2) \in \mathbb{C}^*\times\mathbb{C}^*$, there exists some $g \in G$ such that $(t_1x_1, t_2x_2) = (\mathrm{Ad}(g)x_1, \mathrm{Ad}(g)x_2)$.

It is shown in [Gin00, Theorem 1.2] that for every principal nilpotent pair (e_1, e_2), there exists an associated semisimple pair $\mathbf{h} = (h_1, h_2)$ such that $\mathbf{h}$ is regular and $[h_i, e_j] = \delta_{ij} e_j$ for $i, j = 1, 2$.

The adjoint action of (h_1, h_2) decomposes $\mathfrak{g}$ into weight spaces $\mathfrak{g} = \oplus_{a,b \in \mathbb{Z}^2} \mathfrak{g}_{a,b}$ such that $\mathrm{ad}(h_1)x = ax$ and $\mathrm{ad}(h_2)x = bx$ for all $x \in \mathfrak{g}_{a,b}$.

For every fixed principal nilpotent pair $\mathbf{e}$ with associated semisimple pair $\mathbf{h}$, let $\rho : \mathbb{C}^* \times \mathbb{C}^* \to G$ be the 2-parameter subgroup with differential at the identity being $\mathbb{C}^2 \to \mathfrak{g}$: $(1, 0) \mapsto h_1$, $(0, 1) \mapsto h_2$. We define a $\mathbb{C}^* \times \mathbb{C}^*$ action on $\mathfrak{g}$ by $\mathrm{Ad}(\rho)$ and a $\mathbb{C}^* \times \mathbb{C}^*$ action $\mathfrak{g} \times \mathfrak{g}$ by

$$(t_1, t_2)(x, y) = (t_1^{-1}\mathrm{Ad}(\rho(t_1, t_2))x, t_2^{-1}\mathrm{Ad}(\rho(t_1, t_2))y) \tag{36}$$

such that $\mathbf{e}$ is a fixed point under this action. Note that under these actions, $(\mathfrak{g} \oplus \mathfrak{g})_{a,b} = \mathfrak{g}_{a+1,b} \oplus \mathfrak{g}_{a,b+1}$.

4.3.2. *Stalks of the cuspidal DG module.* In the case of $\mathfrak{sl}_n$, principal nilpotent pairs up to conjucation are in bijection with partitions of n. Indeed, for a partition λ, let e_1 be the associated Jordan normal form and e_2 be the Jordan normal form associated to the transpose λ^t. Then it is easy to check that (e_1, e_2) defines a principal nilpotent pair and all principal nilpotent pairs can be constructed in this way. For $\lambda \vdash n$, let $\mathbf{e}_\lambda$ denote the corresponding principal nilpotent pair up to conjugacy.

The $\mathbb{C}^* \times \mathbb{C}^*$-action defined in Section 4.3.1 induces a $\mathbb{C}^* \times \mathbb{C}^*$-action on $(Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda}$. Consider this action versus the bigrading on $GS(\mathrm{eL}_c)|_{I_\lambda}$ induced by F^{H} and the Euler field $\mathrm{h}_c \in \mathrm{H}_c$. Because the $\mathrm{Ad}(\rho)$-action is lost when taking descent, one has that

Lemma 4.4. *There is a bigraded isomorphism between vector spaces* $(Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda} \cong GS(\mathrm{eL}_c)|_{I_\lambda}$.

Definition 4.5. *We call a Young tableau an almost standard Young tableau (ASYT) if the labels increase rightwards on rows and upwards on columns, with the exception that the labels are allowed to decrease up to* 1 *going up.*

For an example, see Appendix B.2.2 for a full list of all ASYT of three boxes. Recall that if the labels increase rightwards on rows and upwards on columns, we obtain a standard Young tableau. Let ASYT_λ, resp. SYT_λ, be the set of almost standard Young tableaux, resp. standard Young tableaux, of shape λ.

Remark 4.6. *Almost standard Young tableaux appear in the discussion of the "eccentric correspondence" in* [Neg15b, 4.5] *and* [GN24, 2.3]*, which is invented to study the shuffle generators defined by eq. (46). When specialized, the eccentric correspondence captures the geometry of the cuspidal DG algebra* $\mathcal{A}$ *at homological degree* 0.

Since $\mathbf{h}$ is regular, the Borel subalgebras containing both h_1 and h_2 are in bijection with the Weyl group. We fix such a bijection $w \leftrightarrow \mathfrak{b}_w$. Write $\mathfrak{Z} := G \times^B (\mathfrak{n} \times \mathfrak{m})$ and $\mathfrak{n}_w = [\mathfrak{b}_w, \mathfrak{b}_w]$ for $w \in W$. We use the subscript $_r$ to indicate the regular locus, i.e., when the stablizer of the G-action is of the minimal possible dimension. Then $\mathfrak{Z}_r^A = \sqcup_{w \in W} \mathfrak{Z}_r^{w,A}$ with

$$\mathfrak{Z}_r^{w,A} := \{\mathfrak{b}_w\} \times (\mathfrak{n}_w \oplus \mathfrak{m}_w) \cap (\mathfrak{g} \oplus \mathfrak{g})_r^A.$$

Similar to [BG13, Lemma 4.4.1], we have that

Lemma 4.7. *For* $\mathbf{e} = (e_1, e_2)$ *with associated principal semisimple pair* (h_1, h_2)*, the set of Borel subalgebras that contains* e_1, e_2, h_1, h_2 *are in bijection with* ASYT_λ.

We will call such Borel subalgebras almost adapted.

Though $[\mathfrak{n}_w, \mathfrak{m}_w] = \mathfrak{b}_w$, we will insist on writing $[\mathfrak{n}_w, \mathfrak{m}_w]$ to emphasize the A-action on it is induced by composing (36) with $[-, -]$. Note that $\mathfrak{g}_{1,1} = [\mathfrak{n}_w, \mathfrak{m}_w]^A$. We fix an A-stable subspace $\mathfrak{R}_w \subset [\mathfrak{n}_w, \mathfrak{m}_w]$ such that $[\mathfrak{n}_w, \mathfrak{m}_w] = \mathfrak{R}_w \oplus \mathfrak{g}_{1,1}$.

For any bigraded vector space V, denote $\lambda(V)=\sum_{i=0}^{\dim(V)}(-1)^i\mathrm{ch}_A(\wedge^i V)$. Following [BG13, 4.3], we adopt the Ω-notation by setting

$$\Omega(\sum_{i,j} a_{i,j}q^it^j)=\prod(1-q^it^j)^{a_{i,j}} \tag{37}$$

and $\Omega^0(F)=\Omega(F-a_{0,0})$. Also, we write

$$\omega(x)=\frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}$$

Analogous to [BG13, Theorem 4.5.1], we have

Proposition 4.8. *The bigraded character of the stalk $(Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda}$ is*

$$\mathrm{ch}_A((Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})=g_\lambda\frac{(1-qt)^{n-1}}{(1-t)^{n-1}(-t)^{n-1}}\sum_{\sigma\in\mathrm{ASYT}_{\mathbf{e}}}\frac{\Xi_\sigma\prod_{i=1}^n\chi_{n-i+1}^{\mu_{\lfloor c\rfloor}(i)}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})} \tag{38}$$

where

$$\Xi_\sigma:=\hat{\prod}_i\frac{1}{(1-\chi_i^{-1})}\hat{\prod}_{1\le i<j\le n}\omega(\frac{\chi_i}{\chi_j}) \tag{39}$$

The "restricted" product $\hat{\prod}$ means we ignore all the zero linear denominators.

Proof. Denote the commuting variety $\{(x,y)\in\mathfrak{g}\times\mathfrak{g}|[x,y]=0\}_{\mathrm{red}}$ by $\mathfrak{C}$. Let $\mathfrak{C}_r$ denote the regular locus in $\mathfrak{C}$.

By [BG13, Proposition 3.8.6],

$$\mathrm{ch}_A((Rp_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})=\lambda((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}_\lambda})\cdot\sum_{\mathfrak{b}_w\text{ almost adapted}}\lambda((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)_{\mathbf{e}_\lambda})^{-1}\lambda(\mathfrak{R}_w^*)\prod_{i=1}^n\chi_{n-i+1}^{\mu_{\lceil c\rceil}(i)}.$$

One has that $\lambda(\mathfrak{R}_w)=\lambda(([\mathfrak{n}_w,\mathfrak{m}_w]/\mathfrak{g}_{1,1})^*)$. Moreover, by [BG13, Lemma 3.9.1],

$$\lambda((T^*_{\mathfrak{C}_r^A}\mathfrak{C}_r)|_{\mathbf{e}_\lambda})=g_\lambda\cdot\lambda((\mathfrak{g}/(\mathrm{Stab}(e)\oplus\mathfrak{h}))^*)$$
$$\lambda((T^*_{\mathfrak{Z}_r^{w,A}}\mathfrak{Z}_r)_{\mathbf{e}_\lambda})=\lambda(\mathfrak{n}_w)\oplus\lambda\Big(\big((\mathfrak{n}_w\oplus\mathfrak{m}_w)/(\mathfrak{g}\oplus\mathfrak{g})^A\cap(\mathfrak{n}_w\oplus\mathfrak{m}_w)\big)^*\Big)$$

Let R, resp. R^+, denote the set of all roots, resp. positive roots with respect to $\mathfrak{b}_w$. Also recall that the weight of the box x labeled i is defined by $\chi_i = q^{a'(x)}t^{l'(x)}$. We have that

$$
\begin{aligned}
\mathrm{ch}_A(\mathfrak{h}^*) =& n-1\\
\mathrm{ch}_A(\mathfrak{g}^*) =& n-1+\sum_{\alpha_1,\alpha_2\in R} q^{\alpha_1(h_1)}t^{\alpha_2(h_2)} = n-1+\sum_{1\leq i\neq j\leq n}\chi_i\chi_j^{-1}\\
\mathrm{ch}_A(\mathrm{Stab}(e)^*) =& \sum_{(a,b)\in YT_\lambda} q^{-a}t^{-b} = \sum_{i=1}^{n}\chi_i^{-1}\\
\mathrm{ch}_A(\mathfrak{n}_w) =& \sum_{\alpha\in R^+} q^{-\alpha(h_1)}t^{-\alpha(h_2)} = \sum_{1\leq i<j\leq n}\chi_i^{-1}\chi_j\\
\mathrm{ch}_A(\mathfrak{n}_w^*\oplus\{0\}) =& \sum_{\alpha\in R^+} q^{1+\alpha_1(h_1)}t^{\alpha_2(h_2)} = q\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1}\\
\mathrm{ch}_A(\{0\}\oplus\mathfrak{m}_w^*) =& \sum_{\alpha\in R^+} q^{\alpha(h_1)}t^{1+\alpha(h_2)} + (n-1)t + \sum_{\alpha\in R^+,\text{ simple}} q^{-\alpha(h_1)}t^{1-\alpha(h_2)}\\
=&(n-1)t + t\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1} + t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1}\\
\chi([\mathfrak{n}_w,\mathfrak{m}]) =& (n-1)qt + \sum_{\alpha\in R^+} q^{1+\alpha_1(h_1)}t^{1+\alpha_2(h_2)} = (n-1)qt + qt\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1}
\end{aligned}
$$

Since $\lambda(V)=\Omega(\mathrm{ch}_A(V))$ and $\lambda(V/V^A)=\Omega^0(\mathrm{ch}_A(V))$, we further deduce that

$$
\lambda((\mathfrak{g}/(\mathrm{Stab}(e)\oplus\mathfrak{h}))^*)\cdot\Big(\lambda(\mathfrak{n}_w)\oplus\lambda(\big((\mathfrak{n}_w\oplus\mathfrak{m}_w)/(\mathfrak{g}\oplus\mathfrak{g})^A\cap(\mathfrak{n}_w\oplus\mathfrak{m}_w)\big)^*)\Big)^{-1}
$$

$$
\begin{aligned}
=&\Omega(\sum_{1\leq i\neq j\leq n}\chi_i\chi_j^{-1})\Omega((n-1)qt+qt\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1})\\
&\Big(\Omega(\sum_{i=1}^{n}\chi_i^{-1})\Omega(\sum_{1\leq i<j\leq n}\chi_i^{-1}\chi_j)\Omega^0(q\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1})\Big)^{-1}\\
&\Big(\Omega((n-1)t)\Omega^0(t\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1})\Omega^0(t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1})\Big)^{-1}\\
=&(\frac{1-qt}{1-t})^{n-1}\Omega^0((1+qt-q-t)\sum_{1\leq i<j\leq n}\chi_i\chi_j^{-1})\Omega(\sum_{i=1}^{n}\chi_i^{-1})^{-1}\Omega^0(t\sum_{i=1}^{n-1}\chi_{i+1}\chi_i^{-1})^{-1}\\
=&(\frac{1-qt}{1-t})^{n-1}\sum_{\sigma\in\mathrm{ASYT}_{\mathbf{e}}}\frac{\hat{\prod}_{1\leq i<j\leq n}\omega(\frac{\chi_i}{\chi_j})}{\hat{\prod}_i(1-\chi_i^{-1})\hat{\prod}_{i=1}^{n-1}(1-t\frac{\chi_{i+1}}{\chi_i})}\\
=&\frac{(1-qt)^{n-1}}{(1-t)^{n-1}(-t)^{n-1}}\sum_{\sigma\in\mathrm{ASYT}_{\mathbf{e}}}\frac{\chi_1/\chi_n\Xi_\sigma}{\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})}
\end{aligned}
$$

The proposition now follows from the equality

$$
(\lceil\mu_c\rceil(1),\cdots,\lceil\mu_c\rceil(n)) = (\lfloor\mu_c\rfloor(1),\cdots,\lfloor\mu_c\rfloor(n)) + (1,0,\cdots,0,-1). \qquad \square
$$

5. Catalan DG modules and shuffle generators

5.1. **Catalan DG modules.** Following Section 4.1 closely, we define an analogue of the cuspidal DG module. On $\mathcal{B}$ we also have the vector bundle $\underline{[\mathfrak{n},\mathfrak{n}]^*}$ (resp. $\underline{[\mathfrak{n},\mathfrak{n}]}$) whose total space equals $G\times^B[\mathfrak{n},\mathfrak{n}]^*$ (resp. $G\times^B[\mathfrak{n},\mathfrak{n}]$). Let $\pi_{[\mathfrak{n},\mathfrak{n}]}:G\times^B[\mathfrak{n},\mathfrak{n}]\to\mathcal{B}$ be the projection and $\iota_{[\mathfrak{n},\mathfrak{n}]}:\mathcal{B}\to G\times^B[\mathfrak{n},\mathfrak{n}]$ be the zero section. The Koszul complex $(\wedge^\bullet\pi^*_{[\mathfrak{n},\mathfrak{n}]}\underline{[\mathfrak{n},\mathfrak{n}]^*},\partial_{[\mathfrak{n},\mathfrak{n}]})$, with differential given by contraction with the canonical section of $\pi^*_{[\mathfrak{n},\mathfrak{n}]}\underline{[\mathfrak{n},\mathfrak{n}]}$, is quasi-isomorphic to $(\iota_{[\mathfrak{n},\mathfrak{n}]})_*\mathcal{O}_{\mathcal{B}}$. Put $q_{[\mathfrak{n},\mathfrak{n}]}:G\times^B(\mathfrak{n}\times\mathfrak{n})\to G\times^B[\mathfrak{n},\mathfrak{n}]$ by $(g,x,y)\mapsto(g,[x,y])$.

We define

$$\mathcal{A}'_c:=(\pi_{G\times^B(\mathfrak{n}\times\mathfrak{n})\to\mathcal{B}})^*\mathcal{L}_{\lfloor\mu_c\rfloor}\otimes\big(\wedge^\bullet(\pi_{[\mathfrak{n},\mathfrak{n}]}\circ q_{[\mathfrak{n},\mathfrak{n}]})^*\underline{([\mathfrak{n},\mathfrak{n}])^*},q^*_{[\mathfrak{n},\mathfrak{n}]}\partial_{[\mathfrak{n},\mathfrak{n}]}\big).\tag{40}$$

and call it the Catalan DG module at slope c.

Warning: Note that the definitions (30) and (40) use different line bundles.

Write $p':G\times^B(\mathfrak{n}\times\mathfrak{n})\to\mathfrak{g}\times\mathfrak{g}:(g,x,y)\mapsto(g\cdot x,g\cdot y)$. The $\mathbb{C}^*\times\mathbb{C}^*$-actions in Section 4.3.1 induce a $\mathbb{C}^*\times\mathbb{C}^*$-action on $(\mathrm{R}p'_*\mathcal{A}'_c)|_{\mathbf{e}}$. Similar to Proposition 4.8, we have that

Proposition 5.1. *The bigraded character of the stalk* $\mathrm{R}(p_{G\times^B(\mathfrak{n}\times\mathfrak{n})})_*\mathcal{A}'_c)|_{\mathbf{e}}$ *is given by*

$$\mathrm{ch}_A((\mathrm{R}p'_*\mathcal{A}'_c)|_{\mathbf{e}_\lambda})=g_\lambda\sum_{\sigma\in\mathrm{SYT}_\lambda}\frac{\Xi_\sigma\prod_{i=1}^n\chi_i^{\mu_c(n-i+1)}}{\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})}\tag{41}$$

where Ξ_σ *is as defined in (39).*

Remark 5.2. [BG13, Theorem 4.5.1] *states that the* q,t*-character of the stalk of the rank* $n!$ *vector bundle on the commuting variety at* $\mathbf{e}_\lambda$ *is*

$$\frac{g_\lambda}{(1-q)^n(1-t)^n}\sum_{\sigma\in\mathrm{SYT}_\lambda}\Xi_\sigma.\tag{42}$$

As a comparison, the DG algebra in [Gin12] *is defined by pulling* $(\wedge^\bullet\pi^*_{\mathfrak{n}}\underline{\mathfrak{n}^*},\partial_{\mathfrak{n}})$ *back to* $G\times^B(\mathfrak{b}\times\mathfrak{b})$ *and the Catalan DG algebra is defined by pulling* $(\wedge^\bullet\pi^*_{[\mathfrak{n},\mathfrak{n}]}\underline{[\mathfrak{n},\mathfrak{n}]^*},\partial_{[\mathfrak{n},\mathfrak{n}]})$ *back to* $G\times^B(\mathfrak{n}\times\mathfrak{n})$.

Compare (42) and (41) when $\prod_{i=1}^n\chi_i^{\mu_c(n-i+1)}=1$. *The differences lie in the terms* $(1-q)^n(1-t)^n$ *and* $\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})$.

The term $(1-q)^n(1-t)^n$ *arises from the distinction between the support being* $\mathfrak{b}\times\mathfrak{b}$ *versus* $\mathfrak{n}\times\mathfrak{n}$ *(in* $\mathfrak{gl}_n$*). The term* $\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})$ *results from the complex being defined by* $\mathfrak{n}^*$ *versus* $[\mathfrak{n},\mathfrak{n}]^*$.

For a $\mathbb{C}^*\times\mathbb{C}^*$-module $\mathcal{F}$, we use the notation $q^at^b\mathcal{F}$ to shift the original action by the weight (a,b). We will prove in the next sections that the descents of $Rp'_*\mathcal{A}'_c$ and $q^{1-n}Rp_*\mathcal{A}_c$ correspond to the same equivariant K-theory classes on Hilb^n. We conjecture that

Conjecture 5.3. *There exists a* $\mathrm{GL}_n\times\mathbb{C}^*\times\mathbb{C}^*$*-equivariant isomorphism:*

$$Rp'_*\mathcal{A}'_c\cong q^{1-n}Rp_*\mathcal{A}_c.$$

One should note that a priori it is not clear whether $Rp'_*\mathcal{A}'_c$ is concentrated in one degree. In contrast, the sheaf $Rp_*\mathcal{A}_c$ is automatically concentrated in one degree as it is the associated graded of $\mathbf{N}_c$.

5.2. **Cuspidal vs Catalan.**

5.2.1. *Shuffle algebras.* Define

$$K = \mathbb{C}(q,t)(z_1, z_2, \cdots)^{S_\infty}.$$

We endow K with a $\mathbb{C}(q,t)$-algebra structure via the shuffle product

$$f(z_1,\cdots,z_k) * g(z_1,\cdots,z_\ell) = \frac{1}{k!\ell!}\mathrm{Sym}\big[f(z_1,\cdots,z_k)g(z_{k+1},\cdots,z_{k+\ell})\prod_{i=1}^{k}\prod_{j=k+1}^{k+\ell}\omega(\frac{z_i}{z_j})\big].$$

Here Sym denotes symmetrization.

Definition 5.4. *The shuffle algebra $\mathfrak{A}$ is defined as the subspace of K consisting of rational functions in the form of*

$$F(z_1,\cdots,z_k) = \frac{f(z_1,\cdots,z_k)\prod_{1\le i<j\le k}(z_i-z_j)^2}{\prod_{1\le i\ne j\le k}(z_i-qz_j)(z_i-tz_j)}$$

such that f is a symmetric Laurent series satisfying the wheel conditions:

$$f(z_1,z_2,z_3,\dots) = 0 \text{ if } \{\frac{z_1}{z_2},\frac{z_2}{z_3},\frac{z_3}{z_1}\} = \{q,t,\frac{1}{qt}\}.$$

It is shown in [SV13] that there is an isomorphism between $\mathfrak{A}$ and the positive half of the elliptic Hall algebra. Moreover,

Theorem 5.5. *([FT11, SV13]) There exists a geometric action of the algebra $\mathfrak{A}$ on the vector space $\bigoplus_{n\ge0} K^A(\mathrm{Hilb}^n)\otimes_{\mathbb{C}[q^\pm,t^\pm]}\mathbb{C}(q,t)$.*

5.2.2. *Shuffle generators.* Following [Neg22], we define[1]

$$P_{n,m} = \mathrm{Sym}\Big(\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic\rfloor-\lfloor(i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})}\prod_{1\le i<j\le n}\omega(\frac{z_i}{z_j})\Big) \tag{43}$$

According to [BS12], $P_{n,m}$ with $n\ge1$, $m\in\mathbb{Z}$ generate the shuffle algebra $\mathfrak{A}$.

By [Neg22, (2.34) and (2.35)], (43) equals

$$P_{n,m} = \Big(\frac{(1-qt)}{(1-t)(-qt)}\Big)^{n-1}\mathrm{Sym}\Big(\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic\rfloor-\lfloor(i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})}\prod_{1\le i<j\le n}\omega(\frac{z_i}{z_j})\Big). \tag{44}$$

Proposition 5.6. *([Neg15a, Proposition 5.5], [GN15, (49)])Under the action in Theorem 5.5,*

$$P_{n,m}\cdot 1 = \Big(\frac{(1-q)(1-t)}{(1-qt)}\Big)^n\sum_{\lambda\vdash n}\frac{\widetilde{H}_\lambda}{g_\lambda}\sum_{\sigma\in\mathrm{SYT}_\lambda}\frac{\prod_{i=1}^n\chi_{n-i+1}^{\lfloor ic\rfloor-\lfloor(i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})}\Theta_\sigma \tag{45}$$

$$= \frac{(1-t)(1-q)^{n-1}}{(1-qt)(-qt)^{n-1}}\sum_{\lambda\vdash n}\frac{\widetilde{H}_\lambda}{g_\lambda}\sum_{\sigma\in\mathrm{ASYT}_\lambda}\frac{\prod_{i=1}^n\chi_{n-i+1}^{\lfloor ic\rfloor-\lfloor(i-1)c\rfloor}}{\prod_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})}\Theta_\sigma \tag{46}$$

where

$$\Theta_\sigma := \prod_{i=1}^n(1-qt\chi_i)\prod_{1\le i<j\le n}\omega^{-1}(\frac{\chi_j}{\chi_i}).$$

[1]Note that the $P_{m,n}$ here and in [Neg22] is denoted by $\widetilde{P}_{m,n}$ in [GN15], as a certain modification of the $P_{m,n}$ in [Neg14].

Proof. The first identity is exactly [Neg15a, Proposition 5.5] (see also [GN15, (49)]). We show the second identity closely following the proof of *loc. cit.*

As in the proof of [Neg15a, Proposition 5.5] (see also [GN15, (42)]), the localization formula (35) and [Neg15a, Theorem 4.7] imply that $P_{n,m} \cdot 1$ equals

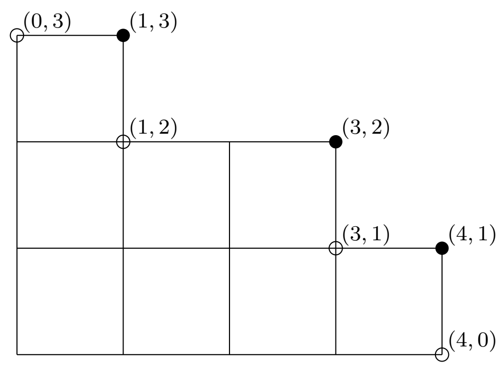
$$\gamma^{n-1} \sum_{\mu \vdash n} \frac{[I_\mu]}{g_\mu} \int \frac{\prod_{i=1}^{n} z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \prod_{1 \leq i < j \leq n} \omega(\frac{z_i}{z_j}) \prod_{\square \in \lambda} \prod_{i=1}^{n} \left(\omega^{-1}(\frac{z_i}{\chi(\square)})(1 - qtz_i) \frac{dz_i}{2\pi i z_i} \right) \Bigg] \tag{47}$$

where $\gamma = \frac{(1-qt)}{(1-t)(-qt)}$ and the integral is taken along contours separating the poles of the function to be integraded.

For a partition $\lambda \vdash n$, [Neg15a, (5.4)] (proved in detail in [BG13, Lemma 4.8.5])

$$\prod_{\square \in \lambda} \left(\omega^{-1}(\frac{z}{\chi(\square)})(1 - qtz) \right) = \frac{\prod_{\square \text{ inner corner of } \mu} (1 - \frac{qtz}{\chi(\square)})}{\prod_{\square \text{ outer corner of } \mu} (1 - \frac{qtz}{\chi(\square)})}.$$

where the notion of inner/outer corners is illustrated below, with hollow circles indicating the inner corners of the partition and the solid circles indicating the outer corners.



[Neg15a, Figure 5.1]

When we integrate over z_n, we pick up a residue whenever qtz_n equals to the weight of some outer corner of the partition λ. Label the box adjacent to this corner by n. Then by change of variables the residue we pick up is the weight of the box labeled by $n-1$ and the integral in (47) becomes

$$\chi_n \int \left[\frac{\prod_{i=1}^{n} z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1} (1 - \frac{z_i}{tz_{i+1}})} \hat{\prod}_{1 \leq i < j \leq n} \omega(\frac{z_i}{z_j}) \hat{\prod}_{\square \in \lambda} \hat{\prod}_{i=1}^{n} \left(\omega^{-1}(\frac{z_i}{\chi(\square)})(1 - qtz_i) \frac{dz_i}{2\pi i z_i} \right) \right] |_{z_n = \chi_n}$$

Next, when we integrate over z_{n-1}, we pick up a residue whenever

- qtz_{n-1} equals the weight of some outer corner of λ. (z_{n-1} equals to the weight of some inner corner of λ)
- $q\frac{z_{n-1}}{\chi_n} = 1$ or $t\frac{z_{n-1}}{\chi_n} = 1$. ($z_{n-1}$ equals to the weight of some box to the left or below the box labeled by n in the last step.)
- $\frac{z_{n-1}}{t\chi_n} = 1$. ($z_{n-1}$ equals to the weight of some box above the box labeled by n in the last step.)

These three conditions together with the condition for z_n exactly define all the possible relative positions between the box labeled by n and the box labeled by $n-1$ in an ASYT such that the box labeled by n is in a corner. One can argue similarly starting from any label on a corner.

Repeating this procedure, we conclude that the integral in (47) equals

$$\sum_{\text{ASYT}}\prod_{i=1}^{n}\chi_i\Big[\frac{\prod_{i=1}^{n} z_{n-i+1}^{\lfloor ic\rfloor-\lfloor (i-1)c\rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{z_i}{tz_{i+1}})}\hat{\prod}_{1\le i<j\le n}\omega(\frac{z_i}{z_j})\hat{\prod}_{\square\in\lambda}\hat{\prod}_{i=1}^{n}\Big(\omega^{-1}(\frac{z_i}{\chi(\square)})(1-qtz_i)\frac{1}{z_i}\Big)\Big]|_{z_i=\chi_i}$$

$$=\Big(\frac{(1-q)(1-t)}{(1-qt)}\Big)^n\sum_{\text{ASYT}_\lambda}\Big[\frac{\prod_{i=1}^{n}\chi_{n-i+1}^{\lfloor ic\rfloor-\lfloor (i-1)c\rfloor}}{\hat{\prod}_{i=1}^{n-1}(1-\frac{\chi_i}{t\chi_{i+1}})}\hat{\prod}_{1\le i<j\le n}\omega^{-1}(\frac{\chi_j}{\chi_i})\prod_{i=1}^{n}(1-qt\chi_i)\Big].$$

The factor $\big(\frac{(1-q)(1-t)}{(1-qt)}\big)^n$ comes from $\hat{\prod}_{i=1}^{n}\omega^{-1}(\frac{\chi_i}{\chi_i})$. □

5.3. Cuspidal = Catalan via shuffle generators.

Proposition 5.7.

$$(P_{n,m}\cdot 1)=\sum_{\lambda\vdash n}\mathrm{ch}_{q,t}((\mathrm{R}p_*\mathcal{A}_c')|_{\mathbf{e}_\lambda})\frac{\widetilde{H}_\lambda}{g_\lambda}$$
$$=q^{1-n}\sum_{\lambda\vdash n}\mathrm{ch}_{q,t}((\mathrm{R}p_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})\frac{\widetilde{H}_\lambda}{g_\lambda}$$

Proof. Recall the following expression from formulae (45) and (46) of $P_{m,n}\cdot 1$:

$$\Theta_\sigma=\Omega(qt\sum_{i=1}^{n}\chi_i-(1-q)(1-t)\sum_{1\le i<j\le n}\frac{\chi_j}{\chi_i})$$

For a Young tableau σ with n boxes and positive integer $k\le n$, we let $\sigma(k)$ denote the Young sub-tableau consisting of the first k labels in σ. Then $\Theta_\sigma=\prod_{i=1}^{n}\Theta(\sigma(j))$ with

$$\Theta(\sigma(j)):=\Omega(qt\chi_j-(1-q)(1-t)\sum_{1\le i<j}\frac{\chi_j}{\chi_i}).\tag{48}$$

Also recall the following expression from formulae (38) and (41) for $\mathcal{A}_c$ and $\mathcal{A}_c'$:

$$\Xi_\sigma=\Omega(-\sum_{i=1}^{n}\chi_j^{-1}+(1-q)(1-t)\sum_{1\le i<j\le n}\frac{\chi_i}{\chi_j})=\prod_{i=1}^{n}\Xi(\sigma(j))$$

with

$$\Xi(\sigma(j)):=\Omega^0(-\chi_j^{-1}+(1-q)(1-t)\sum_{1\le i<j}\frac{\chi_i}{\chi_j}).$$

In view of Propositions 4.8, 5.1 and 5.6, it suffices to show that for all $i=1,\dots,n$,

$$\Theta(\sigma(i))=\Xi(\sigma(i))\frac{g_{\sigma(i)}}{g_{\sigma(i-1)}}\frac{1-qt}{(1-q)(1-t)}.\tag{49}$$

We consider the case $i=n$; the proof in the general case is similar.

We let R_n (resp. C_n) denote the set of boxes of σ in the same row (resp. same column) as $\sigma\setminus\sigma(n-1)$ (excluding $\sigma\setminus\sigma(n-1)$). For $x\in\sigma$ we write $a_k(x),\ell_k(x)$ to indicate the arm and leg of x in $\sigma(k)$. Then we have that

$$\frac{g_{\sigma(n)}}{g_{\sigma(n-1)}}=(1-q)(1-t)\prod_{x\in C_n\cup R_n}\frac{1-q^{1+a_n(x)}t^{-\ell_n(x)}}{1-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}}\cdot\frac{1-q^{-a_n(x)}t^{\ell_n(x)+1}}{1-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}}.$$

Moreover, it is shown in [BG13, 4.10] that

$$\Xi(\sigma(n))=\prod_{x\in C_n}\frac{1-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}}{1-q^{1+a_n(x)}t^{-\ell_n(x)}}\prod_{x\in R_n}\frac{1-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}}{1-q^{-a_n(x)}t^{\ell_n(x)+1}}.$$

Therefore the right hand side of (49) equals

$$(1-qt)\prod_{x\in R_n}\frac{1-q^{1+a_n(x)}t^{-\ell_n(x)}}{1-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}}\prod_{x\in C_n}\frac{1-q^{-a_n(x)}t^{\ell_n(x)+1}}{1-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}}$$

which can be also written as Ω applied to

(50)
$$qt+\sum_{x\in R_n}\left(q^{1+a_n(x)}t^{-\ell_n(x)}-q^{1+a_{n-1}(x)}t^{-\ell_{n-1}(x)}\right)+\sum_{x\in C_n}\left(q^{-a_n(x)}t^{\ell_n(x)+1}-q^{-a_{n-1}(x)}t^{\ell_{n-1}(x)+1}\right)$$

Assume the box labeled by n is at (c,r) and the partition is $(\lambda_1,\cdots,\lambda_{\ell(\lambda)})$ with $\lambda_1\geq\cdots\geq\lambda_{\ell(\lambda)}$. By the proof of [BG13, Lemma 4.10.2], we have that (50) equals

$$\begin{aligned}&qt+\sum_{i=1}^{c-1}(q^{c-i+1}-q^{c-i})t^{r+1-\lambda_{i+1}}+\sum_{j=0}^{r-1}q^{-\lambda_{j+1}+c+1}(t^{r-j+1}-t^{r-j})\\ =&qt+q^{c+1}t^{r+1}\Big(\sum_{i=1}^{c}(q^{-i+1}-q^{-i})t^{-\lambda_i}+\sum_{j=1}^{r}q^{-\lambda_j}(t^{-j+1}-t^{-j})\Big).\end{aligned} \tag{51}$$

By [BG13, Lemma 4.8.5]

$$\sum_{i=1}^{c}(q^{-i+1}-q^{-i})t^{-\lambda_i}+\sum_{j=1}^{r}q^{-\lambda_j}(t^{-j+1}-t^{-j})=-(1-q^{-1})(1-t^{-1})B_{\sigma\backslash\{(c,r)\}}(q^{-1},t^{-1})+1-q^ct^{-r}.$$

Here $B_\mu=\sum_{(\alpha,\beta)\in\mu}q^\alpha t^\beta$ for any Young diagram μ.

Therefore, (51) equals

$$-q^ct^r(1-q)(1-t)B_{\sigma(n-1)}(q^{-1},t^{-1})+q^{c+1}t^{r+1},$$

which is exactly $qt\chi_n-(1-q)(1-t)\sum_{1\leq i<n}\frac{\chi_n}{\chi_i}$. From the expression (48), we obtain the identity (49) and the proposition follows. ∎

Remark 5.8. *A similar formula appears in* [KT22, Lemma 5.13].

We will express the formula (45) as $P_{m,n}\cdot 1=\sum_{\lambda\vdash n}c^\lambda_{m,n}\widetilde{H}_\lambda$, with

$$c^\lambda_{m,n}=\left(\frac{(1-q)(1-t)}{(1-qt)}\right)^n\sum_{\sigma\in\mathrm{SYT}_\lambda}\frac{\prod_{i=1}^n\chi_{n-i+1}^{\lfloor ic\rfloor-\lfloor(i-1)c\rfloor}}{g_\lambda\prod_{i=1}^{n-1}(1-qt\frac{\chi_i}{\chi_{i+1}})}\Theta_\sigma.$$

According to [GN15, Conjecture 6.1] which is implied by [Mel21, Theorem 5.8],

$$\sum_{\lambda\vdash n}c^\lambda_{m,n}=\sum_D q^{\mu-\mathrm{area}(D)}t^{\mathrm{dinv}(D)}=\sum_D q^{\mathrm{dinv}(D)}t^{\mu-\mathrm{area}(D)} \tag{52}$$

is the q,t-Catalan number. Here $\mu=\frac{(m-1)(n-1)}{2}$. The sums on the middle and right are taken over all $\frac{m}{n}$-Dyck paths D. The area and dinv are two combinatorial statistics associated to each dyck path with nonnegative integer values. In particular, when $\mathrm{area}(D)=0$, $\mathrm{dinv}(D)=\mu$ and when $\mathrm{dinv}(D)=0$, $\mathrm{area}(D)=\mu$. Interested readers can refer to [GN15, 6.2] for definitions.

Corollary 5.9. *With respect to the natural $\mathbb{C}^*\times\mathbb{C}^*$-action on $\mathcal{A}_c$,*

- $\mathrm{ch}_{A\times S_n}\Gamma(\mathrm{Hilb}^n,\mathcal{P}\otimes\mathrm{desc}(Rp_*\mathcal{A}_c\boxtimes\mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}})=q^{n-1}P_{m,n}\cdot 1.$
- $\mathrm{ch}_A\Gamma(\mathrm{Hilb}^n,\mathrm{desc}(Rp_*\mathcal{A}_c\boxtimes\mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}})=q^{n-1}\sum_{\lambda\vdash n}c^\lambda_{m,n}.$

Remark 5.10. *Let $\widetilde{\mathcal{P}}$ be the rank $n!$ vector bundle on $\mathfrak{C}$ as defined in* [Gin12]*. Then one similarly has that*

- $\mathrm{ch}_{A\times S_n}\Gamma(\mathfrak{C}_r,\widetilde{\mathcal{P}}\otimes Rp_*\mathcal{A}_c)^{\overline{G}}=q^{n-1}P_{m,n}\cdot 1.$

- $\mathrm{ch}_A\Gamma(\mathfrak{C}_r, Rp_*\mathcal{A}_c)^{\overline{G}} = q^{n-1}\sum_{\lambda\vdash n} c^\lambda_{m,n}$.

Lemma 5.11. $\Gamma(T^*\mathfrak{G}, \widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c)^{\tau_{-c}(\bar{\mathfrak{g}})} = \Gamma(\widetilde{\mathrm{Hilb}^n}, \widetilde{\mathrm{gr}}^{\mathrm{H}}\overline{\mathbf{N}}_c)^{\tau_{-c}(\bar{\mathfrak{g}})}$.

Proof. Clearly we have the inclusion "$\subset$". Thus we only need to count the dimensions. By Corollary 2.9, the left hand side equals $\widetilde{\mathrm{gr}}^{\mathrm{H}}\mathrm{eL}_c$, which is known to have dimension $\frac{(m+n-1)!}{m!n!}$, the Catalan number. Moreover, by Corollary 5.9 and Proposition 4.2, the dimension of the right hand side equals to $\sum_{\lambda\vdash n} c^\lambda_{m,n}(q=1, t=1)$.

By (52), $\sum_{\lambda\vdash n} c^\lambda_{m,n}(q=1, t=1)$ equals the number of $\frac{m}{n}$-dyck paths, which is known to be $\frac{(m+n-1)!}{m!n!}$. □

Proposition 5.12. *Hodge filtrations are compatible with shift functors, i.e., when $c>1$ the following isomorphism is filtered with respect to the Hodge filtration*

$$\mathrm{eL}_c \cong \mathrm{eH}_c\delta\mathrm{e}_- \otimes_{\mathrm{eH}_{c-1}\mathrm{e}} \mathrm{eL}_{c-1}$$

where the filtration on the right hand side is the tensor product filtration such that H_c is endowed with the order filtration.

Proof. Recall the homomorphism $\phi^k_{\overline{\mathbf{N}}_c}$ from (10). Given the equivalence (b)$\Leftrightarrow$(c) in Proposition 2.10, it suffices to show that $\mathrm{gr}(\phi^k_{\overline{\mathbf{N}}_c})$ is an isomorphism for $k \gg 0$.

We claim that when $k\gg 0$, $\Gamma(T^*\mathfrak{G}, \widetilde{\mathrm{gr}}^{\mathrm{H}}(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}})) = \Gamma(\widetilde{\mathrm{Hilb}^n}, \widetilde{\mathrm{gr}}^{\mathrm{H}}(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}})) = \Gamma(\mathrm{Hilb}, \mathcal{O}_{\mathrm{Hilb}}(k))$.

The first equality is [GGS09, Proposition 7.4]. The second equality follows from the $\overline{G}$-equivariant isomorphism: $\big(\mathcal{O}_{\mathfrak{g}\times\mathfrak{g}} \boxtimes \pi_V^*\mathcal{O}_{\mathbb{P}^{n-1}}(k)\big)|_{\widetilde{\mathrm{Hilb}^n}} \cong \mathcal{O}_{\widetilde{\mathrm{Hilb}^n}}(k)$. Here $\pi_V: V\setminus\{0\}\to\mathbb{P}^{n-1}$ is the quotient map.

As a result of the claim, Lemma 5.11 and Proposition 4.2, when $k\gg 0$ we have that

$$\mathrm{gr}(\mathcal{D}_{-c}(\mathfrak{G})^{\det^{-k}} \otimes_{\mathrm{A}_c} \Gamma(\mathfrak{G}, \mathbf{N}_c)^{\tau_{-c}(\bar{\mathfrak{g}})}) \cong \Gamma\big(\mathrm{Hilb}^n, \mathcal{O}_{\mathrm{Hilb}^n}(k)\otimes \mathrm{desc}((Rp_*\mathcal{A}_c \boxtimes (i_V)_*\mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}})\big)$$

which is isomorphic to

$$\Gamma\big(\widetilde{\mathrm{Hilb}^n}, \big(Rp_*(\mathcal{A}_c\otimes(\pi_{\mathfrak{Y}\to\mathcal{B}})^*\mathcal{L}_{(k,\ldots,k)})\big)\boxtimes(i_V)_*\mathcal{O}_V\big) \cong \Gamma(\widetilde{\mathrm{Hilb}^n}, Rp_*\mathcal{A}_{c+k}\boxtimes(i_V)_*\mathcal{O}_V).$$

Using Lemma 5.11 and Proposition 4.2 again, we see that

$$\Gamma(\widetilde{\mathrm{Hilb}^n}, Rp_*\mathcal{A}_{c+k}\boxtimes(i_V)_*\mathcal{O}_V) \cong \mathrm{gr}^{\mathrm{H}}\Gamma(\mathfrak{G}, \overline{\mathbf{N}}_c)^{\bar{\mathfrak{g}}_{-c-k}}$$

and the proposition follows. □

Corollary 5.12 allows us to extend F^{H} from eL_c to L_c for all $c=\frac{m}{n}>1$ by defining a tensor product filtration on

$$\mathrm{L}_c \cong \mathrm{H}_c\delta\mathrm{e}_- \otimes_{\mathrm{A}_{c-1}} \mathrm{eL}_{c-1}. \tag{53}$$

Theorem 5.13. *(i) The bigraded Frobenius character of L_c with respect to the Hodge filtration and the Euler field h_c is*

$$\mathrm{ch}_{S_n\times\mathbb{C}^*\times\mathbb{C}^*}\mathrm{gr}^{\mathrm{H}}\mathrm{L}_c = (P_{m,n}\cdot 1)(q, q^{-1}t).$$

(ii) In $\mathrm{Coh}^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n)$, we have that

$$GS(\mathrm{eL}_c) \cong q^{1-n}\mathrm{desc}\big((Rp_*\mathcal{A}_c\boxtimes(i_V)_*\mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}}\big).$$

Proof. Because of Proposition 5.12 and Proposition 2.10, we have $GS(\mathrm{eL}_c) = \Psi_c(\overline{\mathbf{N}}_c)$.

By [GS05, Theorem 4.5], $GS(\mathrm{eH}_c) = \mathcal{P}$. This plus (53) implies that as $\mathbb{C}^*\times\mathbb{C}^*$-modules, $\mathrm{gr}^{\mathrm{H}}\mathrm{L}_c = \Gamma(\mathrm{Hilb}^n, \mathcal{P}\otimes\Psi(\overline{\mathbf{N}}_c))$.

By Proposition 4.2,

$$\mathcal{P}\otimes\Psi(\overline{\mathbf{N}}_c) = q^a t^b\big(\mathcal{P}\otimes\mathrm{desc}((Rp_*\mathcal{A}_c\boxtimes(i_V)_*\mathcal{O}_V)|_{\widetilde{\mathrm{Hilb}^n}})\big)$$

whose space of global sections has bigraded Frobenius character $q^{a+n-1}t^b P_{m,n} \cdot 1$ according to Corollary 5.9, for some integers a, b.

As a consequence, $\mathrm{ch}_A(\mathrm{gr}^{\mathrm{H}}\mathrm{eL}_c) = q^{a+n-1}t^b \sum_{\lambda \vdash n} c^{\lambda}_{m,n}(q, q^{-1}t)$. The change of variable $(q, q^{-1}t)$ comes from the fact that the Euler field h_c acts by weight $(1, -1)$.

It remains to show $q^{a+n-1}t^b = 1$. However, the highest, resp. lowest weight of eL_c under the action of h_c is μ, resp. $-\mu$. Given (52) we see that $q^{a+n-1}t^b = 1$ and the theorem follows. □

5.4. **Link homology.** We can now conclude Theorem A from the introduction.

Let $\mathcal{V}$ be the tautological vector bundle on Hilb^n of rank n characterized by $\mathcal{V}|_I = \mathbb{C}[x, y]/I$ for any $I \in \mathrm{Hilb}^n$. Define $\mathcal{O}_{\mathrm{Hilb}^n}(1) = \wedge^n \mathcal{V}$. Denote by $\mathcal{V}_{\mathrm{st}}$ the direct summand of $\mathcal{V}$ such that $\mathcal{V} := \mathcal{O}_{\mathrm{Hilb}^n} \oplus \mathcal{V}_{\mathrm{st}}$. In $\bigoplus_{n\geq 0} K^A(\mathrm{Hilb}^n) \otimes_{\mathbb{C}[q^{\pm}, t^{\pm}]} \mathbb{C}(a, q, t)$, define $\Lambda(\mathcal{V}_{\mathrm{st}}, a) = \bigoplus_{i=0}^{n-1} a^i (\wedge^i \mathcal{V}_{\mathrm{st}})$.

Theorem 5.14. *([Mel22, Corollary 3.4]) Up to a constant factor, the triply graded Euler characteristic $\mathrm{ch}_{a,q,t}(\mathrm{HHH}(T_{m,n}))$ equals the matrix coefficient $\langle \Lambda(\mathcal{V}_{\mathrm{st}}, a) | P_{m,n} | 1 \rangle$.*

As a corollary of Theorem 5.14 and Theorem 5.13, we have

Theorem 5.15. *For $m > n$ and $(m, n) = 1$, there is a triply graded isomorphism when $m > n$:*

$$\bigoplus_{i,j,k} \mathrm{HHH}^{i,j,k}(T_{m,n}) \cong \bigoplus_i \mathrm{Hom}_{S_n}\big(\wedge^i \mathfrak{h}, \bigoplus_{j,k} \mathrm{gr}^{\mathrm{H}}_j\big(\mathrm{L}_{\frac{m}{n}}(k)\big)\big). \tag{54}$$

5.5. **Filtrations.** Since the left hand side of (54) is m, n-symmetric, we see that

Corollary 5.16. *Hodge filtrations on $\mathrm{eL}_{\frac{m}{n}}$ and $\mathrm{eL}_{\frac{n}{m}}$ are compatible with the isomorphism $\mathrm{eL}_{\frac{m}{n}} \cong \mathrm{eL}_{\frac{n}{m}}$.*

Consider a partial order on the positive rational numbers in the following way: for coprime pairs (m, n), $\frac{m}{n} \prec \frac{m+n}{n}$; if $n < m$, then $\frac{m}{n} \prec \frac{n}{m}$.

On can always go from $c = \frac{m}{n}$ to $\frac{1}{n'}$ for some integer $n' > 1$ through a chain of rational numbers decreasing under the order $(\mathbb{Q}_{>0}, \prec)$.

Definition 5.17. [GORS14, Theorem 4.1] *We define a filtration F^{ind} inductively as follows:*

$$0 = F^{\mathrm{ind}}_{-1}\mathrm{eL}_{\frac{1}{n}} \subset F^{\mathrm{ind}}_0\mathrm{eL}_{\frac{1}{n}} = \mathrm{eL}_{\frac{1}{n}} = \mathrm{L}_{\frac{1}{n}}.$$

Next, F^{ind} is defined inductively under the order $(\mathbb{Q}_{>0}, \prec)$ using the isomorphisms:

$$\text{when } m, n > 1, \quad \mathrm{eL}_{\frac{m}{n}} \cong \mathrm{eL}_{\frac{n}{m}} \quad \text{([CEE09, 8.2])}$$
$$\text{when } c > 1, \quad \mathrm{L}_c \cong \mathrm{H}_c\mathrm{e}_- \otimes_{\mathrm{eH}_{c-1}\mathrm{e}} \mathrm{eL}_{c-1} \quad \text{([GS05, Theorem 1.6])}$$

Combining Proposition 5.12 and Corollary 5.16, we obtain that

Proposition 5.18. *$F^{\mathrm{H}}_j\mathrm{L}_c = F^{\mathrm{ind}}_j\mathrm{L}_c$ when $c > 1$ and $F^{\mathrm{H}}_j\mathrm{eL}_c = F^{\mathrm{ind}}_j\mathrm{eL}_c$ when $c > 0$.*

As a corollary, we are able to recover the following first proved by Haiman in [Hai98].

Proposition 5.19. *The punctual Hilbert scheme is Cohen-Macaulay.*

Proof. [2] □

On H_c we have the Fourier transform defined by

$$\Phi_c(x_i) = y_i, \quad \Phi_c(y_i) = -x_i, \quad \Phi_c(w) = w \tag{55}$$

which defines the Dunkl bilinear form

$$(-,-)_c : \mathbb{C}[\mathfrak{h}] \times \mathbb{C}[\mathfrak{h}] \to \mathbb{C}, \quad (f, g)_c = [\Phi_c(f)g]|_{x_i=0}.$$

[2] Write it

When $c = \frac{m}{n} > 0$, with m, n coprime, $(-,-)_c$ has a nonzero kernel I_c and the resulting quotient $\mathbb{C}[\mathfrak{h}]/I_c$ is exactly isomorphic to L_c ([DO03, Proposition 2.34]). Inside $\mathbb{C}[\mathfrak{h}]$, take the ideal $\mathfrak{a} := (\mathbb{C}[\mathfrak{h}]_+^W)$. Let β_c be a highest weight vector in L_c and let $^{\perp_c}$ denote orthogonal complement with respect to $(-,-)_c$.

Definition 5.20. *The algebraic filtration*[3] *is defined by*

$$F_i^{\mathfrak{a}}(\mathrm{L}_c) = \Phi_c[(\mathfrak{a}^{i+1})^{\perp_c}]\beta_c.$$

Proposition 5.21. *For $m > 0$ and $(m,n) = 1$, there is a triply graded isomorphism:*

$$\mathrm{HHH}(T_{m,n}) \cong \mathrm{Hom}_{S_n}\big(\wedge^\bullet \mathfrak{h}, \mathrm{gr}^{\mathfrak{a}}_\bullet(\oplus \mathrm{L}_c(\bullet))\big). \tag{56}$$

Proof. It is shown in [Ma24] that $F^{\mathrm{ind}} = F^{\mathfrak{a}}$. This plus Proposition 5.18 and Corollary 5.15 implies that Proposition 5.21 holds when $m > n$. Denote the right hand side of (56) by $\mathfrak{H}_{m,n}$, it remains to show a triply graded isomorphism $\mathfrak{H}_{m,n} \cong \mathfrak{H}_{n,m}$.

By [Gor13, Corollary 1.1], for all $k \geq 0$ there is an isomorphism

$$\mathrm{Hom}_{S_n}(\wedge^k \mathbb{C}^{n-1}, \mathrm{L}_{\frac{m}{n}}) \cong \mathrm{Hom}_{S_m}(\wedge^k \mathbb{C}^{m-1}, \mathrm{L}_{\frac{n}{m}}). \tag{57}$$

via identifications with spaces of differential forms on a zero-dimensional moduli space associated with the plane curve singularity $x^m = y^n$. By [GORS14, Proposition 1.5], (57) is a bigraded isomorphism with respect to the algebraic filtration and the Euler field h_c. This finishes the proof of the proposition. □

[3] In [Ma24] (and [GORS14]) the filtration $F^{\mathfrak{a}}$ is called the power filtration, while the algebraic filtration is its associated Kazhdan filtration.

Appendix A. Fourier transforms of cuspidal $\mathcal{D}$-modules

In this section, we prove an auxilary result about the Fourier transform of the cuspidal character $\mathcal{D}$-module. Although unrealted to the main results of the paper, this may be of independent interest.

A.1. **Invariance under the Fourier transform.** The map $\mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}$ sending $(x, x^*) \mapsto (x^*, -x)$ induces an isomorphism $\sigma_1 : \mathcal{D}(\mathfrak{g}) \cong \mathcal{D}(\mathfrak{g}^*)$.

Further identitifying $\sigma_2 : \mathcal{D}(\mathfrak{g}^*) \cong \mathcal{D}(\mathfrak{g})$ via a non-degenerate bilinear form $\mathfrak{g} \cong \mathfrak{g}^*$, we have obtain the Fourier transform induced by $\sigma := \sigma_2 \circ \sigma_1$:

$$\mathbb{F} : \mathcal{D}_{\mathfrak{g}}-\text{mod} \to \mathcal{D}_{\mathfrak{g}}-\text{mod}.$$

Proposition A.1. *As $\mathcal{D}_{\mathfrak{g}}$-modules, $\mathbb{F}\mathbf{N}_c \cong \mathbf{N}_c$.*

Proof. Put $\iota : T^*\mathfrak{g} \to T^*\mathfrak{g}, (x, x^*) \mapsto (x^*, -x)$. Then

$$SS(\mathbb{F}(\mathbf{N}_c)) = \iota(SS(\mathbf{N}_c)) \subset \mathcal{N} \times \mathcal{N}.$$

By [Lus87], $\mathbb{F}(\mathbf{N}_c)$ is again a cupidal character $\mathcal{D}$-module and is determined by the monodromy of its restriction to $\mathcal{N}_r$. Therefore the proposition follows from the following claim:

Claim: There is an isomorphism $\iota^\dagger(\mathbb{F}(\mathbf{N}_c)) \cong \mathcal{F}_c$ where $\iota : \mathcal{N}_r \hookrightarrow \mathfrak{g}$.

Recall that the cuspidal character $\mathcal{D}$-module $\mathbf{N}_c$ can be expressed as $\widetilde{\operatorname{Ind}}_B^G \mathbf{L}$. Therefore $\mathbb{F}(\mathbf{N}_c) = \widetilde{\operatorname{Ind}}_B^G \mathbb{F}(\mathbf{L})$.

Since

$$\Gamma(\mathfrak{g}, \mathbf{L}) = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot O(\mathfrak{b}_-) + \sum_i \mathcal{D}(\mathfrak{g})(x_i\partial_{x_i} - ic) + \mathcal{D}(\mathfrak{g}) \cdot S([\mathfrak{n}, \mathfrak{n}])) \tag{58}$$

we see that

$$\Gamma(\mathfrak{g}, \mathbb{F}(\mathbf{L})) = \mathcal{D}(\mathfrak{g})/(\mathcal{D}(\mathfrak{g}) \cdot S(\mathfrak{b}) + \sum_i \mathcal{D}(\mathfrak{g})(\partial_{y_i} y_i + ic) + \mathcal{D}(\mathfrak{g}) \cdot O([\mathfrak{n}_-, \mathfrak{n}_-])). \tag{59}$$

Consider the standard $\mathfrak{sl}_2$-triple E, F, H (defined by (20)). Then (58) and (59) tells us that $\mathbf{L}$ is the minimal extension of a local system supported on $B \cdot E$ and $\mathbb{F}(\mathbf{L})$ is the minimal extension of a local system supported on $B \cdot F$ defined by a horizontal section

$$y_1^{-c} y_2^{-2c} \cdots y_{n-1}^{-(n-1)c}. \tag{60}$$

Moreover, $\mathbf{L}|_{T \cdot E}$ is a local system on $T \cdot E$ such that

$$\mathcal{F}_c = \widetilde{\operatorname{Ind}}_T^G i_\dagger^E (i^E)^\dagger \iota^\dagger \mathbf{L}$$

with $i^E : T \cdot E \to \mathcal{N}_r$. Similarly, $\mathbb{F}(\mathbf{N}_c)|_{T \cdot F}$ is a local system on $T \cdot F$ such that

$$(\mathbb{F}(\mathbf{N}_c))|_{\mathcal{N}_r} = \widetilde{\operatorname{Ind}}_T^G i_\dagger^F (i^F)^\dagger \iota^\dagger \mathbb{F}(\mathbf{L})$$

with $i^F : T \cdot F \to \mathcal{N}_r$.

It suffices to show that $(i^F)^\dagger \mathcal{F}_c = (i^F)^\dagger \iota^\dagger \mathbb{F}(\mathbf{L})$. Recall that the pullback of $\mathcal{F}_c$ along the fibration $q : U \to \mathcal{N}_r$ is $\mathcal{E}_c$ and $\mathcal{E}_c$ is defined by the horizontal section s^c (eq. (5)).

$$s^c|_{T \cdot F} = v_1^{nc} y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c.$$

Therefore, $(i^F)^\dagger \mathcal{F}_c$ is defined by the horizontal section

$$y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c.$$

Finally, the lemma follows from the observation that the functions $y_1^{(n-1)c} y_2^{(n-2)c} \cdots y_{n-1}^c$ and $y_1^{-c-1} y_2^{-2c-1} \cdots y_{n-1}^{-(n-1)c-1}$ (eq. (60)) define the same local system on $T \cdot F$ as

$$((n-1)c, (n-2)c, \cdots, c) - (m, m, \cdots, m) = (-c, -2c, \cdots, -(n-1)c). \qquad \square$$

A.2. **An explicit description of the Fourier transform.** The contents in this subsection were observed by V. Ginzburg.

Let $x = (x_{ij})$ be the standard coordinates of $\mathfrak{gl}_n$ and $(\partial) = (\partial_{x_{ij}})_{1\leq i,j\leq n}$. Take

$$e := \frac{1}{2}\mathrm{tr}(x^2), \quad f := -\frac{1}{2}\mathrm{tr}(\partial^2), \quad h = \sum_{1\leq i,j\leq n} x_{ij}\partial_{x_{ij}} + (n^2-1)/2. \tag{61}$$

Clearly, $[e,f]=h$, $[h,e]=2e$ and $[h,f]=-2f$. Notice that $[e,-],[f,-],[h,-]$ all preserve the homogeneous components of $\mathcal{D}(\mathfrak{g})$ (with $\deg(x_{ij})$=$\deg(\partial_{x_{ij}})$=1). In other words, this $\mathfrak{sl}_2$-action on $\mathcal{D}(\mathfrak{g})$ is locally finite and thus integrable. Moreover, $e-f$ acts on $\mathcal{D}(\mathfrak{g})$ via

$$[e-f, x_{ij}] = [-f, x_{ij}] = \partial_{x_{ij}}, \quad [e-f, \partial_{x_{ij}}] = [e, \partial_{x_{ij}}] = -x_{ij}. \tag{62}$$

Hence its exponential Ad $e^{\frac{i\pi}{2}(e-f)}$ gives exactly the Fourier transform σ.

On the other hand, any $\mathcal{D}(\mathfrak{g})$-module inherits such an $\mathfrak{sl}_2$-action. By [CEE09, example 63], the action of $\{e,f,h\}$ on $\mathbf{N}_c$ is locally finite and thus also integrable. Denote the action of $e^{\frac{i\pi}{2}(e-f)}$ on $\mathbf{N}_c$ by Φ. Then (62) implies that

$$\Phi(x_{ij}a) = \partial_{x_{ij}}\Phi(a), \quad \Phi(\partial_{x_{ij}}a) = -x_{ij}\Phi(a), \quad \forall a\in\mathbf{N}_c, 1\leq i,j\leq n.$$

Therefore, Φ gives an explicit isomorphism between $\mathbf{N}_c$ and $\mathbb{F}(\mathbf{N}_c)$.

Appendix B. Examples

We compile some computations of $\displaystyle\sum_{\lambda\vdash n}\frac{\mathrm{ch}_{q,t}((\mathrm{R}p_*\mathcal{A}_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$ and $\displaystyle\sum_{\lambda\vdash n}\frac{\mathrm{ch}_{q,t}((\mathrm{R}p'_*\mathcal{A}'_c)|_{\mathbf{e}_\lambda})}{g_\lambda}$. In view of [KT22], setting $q\to 1$ these statistics are also Shalika germs in the corresponding cases.

B.1. $n=2$.

B.1.1. *Catalan* $\mathcal{A}'_{k+\frac{1}{2}}$.

$$\overbrace{\frac{q^k}{1-\frac{t}{q}}}^{\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}} + \overbrace{\frac{t^k}{1-\frac{q}{t}}}^{\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}} = \sum_{i=0}^{k} q^i t^{k-i}$$

B.1.2. *Cuspdial* $\mathcal{A}_{k+\frac{1}{2}}$.

$$\overbrace{\frac{q^{k+1}}{1-\frac{t}{q}}}^{\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}} + \overbrace{\frac{t^{k+1}(1-q)(1-qt)}{(1-t)(1-t^2)(1-\frac{q}{t})}}^{\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}} + \overbrace{\frac{t^k(1-qt^2)}{(1-t^2)(1-\frac{1}{t})}}^{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} = q(\sum_{i=0}^{k} q^i t^{k-i})$$

B.2. $n=3$.

B.2.1. *Catalan* $\mathcal{A}'_{\frac{2}{3}}$.

$$\overbrace{\frac{q}{\left(1-\frac{t}{q}\right)\left(1-\frac{q^2}{t}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array}}+\overbrace{\frac{t}{\left(1-\frac{q}{t}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}}+\overbrace{\frac{q}{\left(1-\frac{t}{q}\right)\left(1-\frac{t}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}+\overbrace{\frac{t}{\left(1-\frac{q}{t}\right)\left(1-\frac{q}{t^2}\right)}}^{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}}$$
$$=q+t$$

B.2.2. *Cuspidal* $\mathcal{A}_{\frac{2}{3}}$.

$$\overbrace{\frac{q^3}{\left(1-\frac{t}{q}\right)\left(1-\frac{t}{q^2}\right)}}^{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}}$$
$$+\overbrace{\frac{q\left(1-q^2\right)t(1-qt)}{(1-t)\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}}+\overbrace{\frac{qt(1-qt)^2}{(1-t)^2\left(1-\frac{q}{t}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline\end{array}}+\overbrace{\frac{q\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array}}$$
$$+\overbrace{\frac{(1-q)^2t^3(1-qt)^2}{(1-t)^2\left(1-t^2\right)^2\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)}}^{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}}+\overbrace{\frac{(1-q)t^3(1-qt)\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\left(1-t^2\right)\left(1-t^3\right)\left(1-\frac{q}{t^2}\right)}}^{\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}}$$
$$+\overbrace{\frac{(1-q)t^2(1-qt)\left(1-qt^2\right)}{\left(1-\frac{1}{t}\right)(1-t)\left(1-t^2\right)\left(1-t^3\right)\left(1-\frac{q}{t^2}\right)}}^{\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}}+\overbrace{\frac{t\left(1-qt^2\right)\left(1-qt^3\right)}{\left(1-\frac{1}{t^2}\right)\left(1-\frac{1}{t}\right)\left(1-t^2\right)\left(1-t^3\right)}}^{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}$$
$$=q^2(q+t)$$

B.3. $n=4$.

B.3.1. *Catalan* $\mathcal{A}'_{\frac{3}{4}}$.

$$\overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 4 \\ \hline\end{array}} + \overbrace{\frac{qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline\end{array}}$$

$$+ \overbrace{\frac{(1-q)qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2}}^{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}} + \overbrace{\frac{q^3}{\left(1-\frac{t}{q^3}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 3 \\ \hline\end{array}} + \overbrace{\frac{q^3}{\left(1-\frac{q^3}{t}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}}$$

$$+ \overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)^2\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array}} + \overbrace{\frac{qt}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}}$$

$$+ \overbrace{\frac{qt(1-t)}{\left(1-\frac{q}{t}\right)^2\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}} + \overbrace{\frac{t^3}{\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{q}{t^3}\right)}}^{\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 4 \\ \hline\end{array}} + \overbrace{\frac{t^3}{\left(1-\frac{q}{t^3}\right)\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)}}^{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}}$$

$$= q^3 + q^2 t + qt + qt^2 + t^3$$

References

[Ach] Pramod Achar, *Equivariant mixed Hodge modules*, Lecture notes.

[Ba23] Ana Bălibanu, *The partial compactification of the universal centralizer*, Selecta Math. (N.S.) **29** (2023), no. 5, Paper No. 85, 36.

[BE09] Roman Bezrukavnikov and Pavel Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 397–425.

[BEG03] Yuri Berest, Pavel Etingof, and Victor Ginzburg, *Finite-dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. (2003), no. 19, 1053–1088.

[BFG06] Roman Bezrukavnikov, Michael Finkelberg, and Victor Ginzburg, *Cherednik algebras and Hilbert schemes in characteristic p*, Represent. Theory **10** (2006), 254–298.

[BG98] F Bergeron and G Garsia, *Science fiction and Macdonald's polynomials*, arXiv preprint math/9809128 (1998).

[BG13] Gwyn Bellamy and Victor Ginzburg, *Some combinatorial identities related to commuting varieties and Hilbert schemes*, Math. Ann. **355** (2013), no. 3, 801–847.

[BG15] ———, *Hamiltonian reduction and nearby cycles for mirabolic $\mathscr{D}$-modules*, Adv. Math. **269** (2015), 71–161.

[BL94] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994.

[BO] Roman Bezrukavnikov and Andrei Okounkov, *Monodromy and derived equivalences*, unpublished.

[BS12] Igor Burban and Olivier Schiffmann, *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161** (2012), no. 7, 1171–1231.

[CEE09] Damien Calaque, Benjamin Enriquez, and Pavel Etingof, *Universal KZB equations: the elliptic case*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 165–266.

[CG10] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Ltd., Boston, MA, 2010.

[Che98] Jan Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math. **183** (1998), no. 1, 39–90.

[Del70] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[DO03] C. F. Dunkl and E. M. Opdam, *Dunkl operators for complex reflection groups*, Proc. London Math. Soc. (3) **86** (2003), no. 1, 70–108.

[EG02] Pavel Etingof and Victor Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348.

[EGL15] Pavel Etingof, Eugene Gorsky, and Ivan Losev, *Representations of rational Cherednik algebras with minimal support and torus knots*, Adv. Math. **277** (2015), 124–180.

[EH19] Ben Elias and Matthew Hogancamp, *On the computation of torus link homology*, Compos. Math. **155** (2019), no. 1, 164–205.

[EKLS21] Pavel Etingof, Vasily Krylov, Ivan Losev, and José Simental, *Representations with minimal support for quantized Gieseker varieties*, Math. Z. **298** (2021), no. 3-4, 1593–1621.

[FG10] Michael Finkelberg and Victor Ginzburg, *On mirabolic D-modules*, Int. Math. Res. Not. IMRN (2010), no. 15, 2947–2986.

[FT11] B. L. Feigin and A. I. Tsymbaliuk, *Equivariant K-theory of Hilbert schemes via shuffle algebra*, Kyoto J. Math. **51** (2011), no. 4, 831–854.

[GG06] Wee Liang Gan and Victor Ginzburg, *Almost-commuting variety, $\mathscr{D}$-modules, and Cherednik algebras*, IMRP Int. Math. Res. Pap. (2006), 26439, 1–54.

[GGOR03] Victor Ginzburg, Nicolas Guay, Eric Opdam, and Raphaël Rouquier, *On the category $\mathscr{O}$ for rational Cherednik algebras*, Invent. Math. **154** (2003), no. 3, 617–651.

[GGS09] V. Ginzburg, I. Gordon, and J. T. Stafford, *Differential operators and Cherednik algebras*, Selecta Math. (N.S.) **14** (2009), no. 3-4, 629–666.

[GH96] A. M. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combin. **5** (1996), no. 3, 191–244.

[Gin00] Victor Ginzburg, *Principal nilpotent pairs in a semisimple Lie algebra. I*, Invent. Math. **140** (2000), no. 3, 511–561.

[Gin12] ———, *Isospectral commuting variety, the Harish-Chandra D-module, and principal nilpotent pairs*, Duke Math. J. **161** (2012), no. 11, 2023–2111.

[GN15] Eugene Gorsky and Andrei Neguţ, *Refined knot invariants and Hilbert schemes*, J. Math. Pures Appl. (9) **104** (2015), no. 3, 403–435.

[GN17] ———, *Infinitesimal change of stable basis*, Selecta Math. (N.S.) **23** (2017), no. 3, 1909–1930.

[GN24] ———, *Hecke categories, idempotents, and commuting stacks*, arXiv: 2406.05215.

[Gor12] Iain Gordon, *Macdonald positivity via the Harish-Chandra D-module*, Invent. Math. **187** (2012), no. 3, 637–643.

[Gor13] Evgeny Gorsky, *Arc spaces and DAHA representations*, Selecta Math. (N.S.) **19** (2013), no. 1, 125–140.

[GORS14] Eugene Gorsky, Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende, *Torus knots and the rational DAHA*, Duke Math. J. **163** (2014), no. 14, 2709–2794.

[GS05] I. Gordon and J. T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. **198** (2005), no. 1, 222–274.

[GSV23] Nicolle González, José Simental, and Monica Vazirani, *Higher rank (q,t)-catalan polynomials, affine Springer fibers, and a finite rational shuffle theorem*, arXiv:2303.15694 (2023).

[Hai98] Mark Haiman, *t,q-Catalan numbers and the Hilbert scheme*, Discrete Math. **193** (1998), no. 1-3, 201–224.

[Hai01] ———, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006.

[Hai02] ———, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407.

[Hai03] ———, *Combinatorics, symmetric functions, and Hilbert schemes*, Current developments in mathematics, 2002, Int. Press, Somerville, MA, 2003, pp. 39–111.

[Hog17] Matthew Hogancamp, *Khovanov-rozansky homology and higher Catalan sequences*, arXiv:1704.01562 (2017).

[Hog18] ———, *Categorified Young symmetrizers and stable homology of torus links*, Geom. Topol. **22** (2018), no. 5, 2943–3002.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, japanese ed., Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008.

[Kas03] Masaki Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[Kho07] Mikhail Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, Internat. J. Math. **18** (2007), no. 8, 869–885.

[KT22] Oscar Kivinen and Cheng-Chiang Tsai, *Shalika germs for tamely ramified elements in GL_n*, arXiv preprint arXiv:2209.02509 (2022).

[Lau83] Gérard Laumon, *Sur la catégorie dérivées des d-modules filtrés*, Algebraic geometry, Springer, 1983, pp. 151–237.

[Lus86] George Lusztig, *Character sheaves. IV*, Adv. in Math. **59** (1986), no. 1, 1–63.

[Lus87] ———, *Fourier transforms on a semisimple Lie algebra over $\mathbf{F}_q$*, Algebraic groups Utrecht 1986, Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 177–188.

[Ma24] Xinchun Ma, *Rational cherednik algebras and torus knot invariants*, arXiv:2402.18770 (2024).

[Mel21] Anton Mellit, *Toric braids and (m,n)-parking functions*, Duke Math. J. **170** (2021), no. 18, 4123–4169.

[Mel22] ———, *Homology of torus knots*, Geom. Topol. **26** (2022), no. 1, 47–70.

[Nak99] Hiraku Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.

[Neg14] Andrei Neguţ, *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN (2014), no. 22, 6242–6275.

[Neg15a] ———, *Moduli of flags of sheaves and their K-theory*, Algebr. Geom. **2** (2015), no. 1, 19–43.

[Neg15b] ———, *Quantum Algebras and Cyclic Quiver Varieties*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)–Columbia University.

[Neg22] ———, *W-algebras associated to surfaces*, Proc. Lond. Math. Soc. (3) **124** (2022), no. 5, 601–679.

[OY16] Alexei Oblomkov and Zhiwei Yun, *Geometric representations of graded and rational Cherednik algebras*, Adv. Math. **292** (2016), 601–706.

[Pop18] Mihnea Popa, *$\mathcal{D}$-modules in birational geometry*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 781–806.

[Sai88] Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995.

[Sai90] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[Sai22] Takahiro Saito, *A description of monodromic mixed Hodge modules*, J. Reine Angew. Math. **786** (2022), 107–153.

[Sch] Christian Schnell, *An overview of Morihiko Saito's theory of mixed Hodge modules*, Representation theory, automorphic forms & complex geometry, Int. Press, Somerville, MA, pp. 27–80.

[SV13] Olivier Schiffmann and Eric Vasserot, *The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$*, Duke Math. J. **162** (2013), no. 2, 279–366.

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