

Rational Cherednik Algebras and Torus Knot Invariants

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Abstract

In [GORS14], the HOMFLY polynomial of the (m, n) torus knot $T_{m, n}$ is extracted from the doubly graded character of the finite-dimensional representation $L_{\frac{m}{n}}$ of the type A_{n-1} rational Cherednik algebra. It is furthermore conjectured in *loc. cit.* that one can obtain the triply-graded Khovanov-Rozansky homology of $T_{m, n}$ by considering a certain filtration on $L_{\frac{m}{n}}$. In this paper, we show that two of the proposed candidates, the algebraic filtration and the inductive filtration, are equal.

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1 Introduction

1.1 The rational Cherednik algebra and link homology

The HOMFLY polynomial is a link invariant defined by the skein relation in the form of a two-variable polynomial, generalizing the one-variable Alexander polynomial and the Jones polynomial. In the early 2000s, these link polynomials were realized as Euler characteristics of various link homology theories. This process was inspired by the ideas of “categorification”, for a survey, see [Kho06]. Remarkably, the HOMFLY polynomial was categorified by the Khovanov–Rozansky (KhR) link homology using matrix factorization [KR08]. Soon after, the construction of the KhR homology was modified using Hochschild homology of Soergel bimodules (hence the notation HHH), which revealed the connection between link invariants and Kazhdan-Lusztig theory [Kho07]. The past twenty years have seen a large variety of incarnations of KhR homology, to name a few: the (co)homology of braid/positroid/Richardson varieties, affine/global Springer theory, coherent sheaves on the Hilbert scheme of points and representation theory of double affine Hecke algebras (for a survey, see [GKS23]).

This paper will focus on the representation theoretic model of the KhR homology. Let m, n be a coprime positive integer pair and let \mathfrak{h} denote the $(n - 1)$ -dimensional standard representation of the symmetric group S_n . The rational Cherednik algebra $H_{\frac{m}{n}}$ (also called rational DAHA) is a deformation of $D(\mathfrak{h}) \rtimes S_n$ at the parameter $\frac{m}{n}$ where $D(\mathfrak{h})$ is the ring of differential operators on \mathfrak{h} . The algebra $H_{\frac{m}{n}}$ has a unique finite-dimensional irreducible representation which we denote by $L_{\frac{m}{n}}$. Under the action of the Euler field, $L_{\frac{m}{n}}$ decomposes into a direct sum of eigenspaces, allowing us to define the associated q -graded Euler characteristic $\text{ch}_q(L_{\frac{m}{n}}) = \sum_i \dim(L_{\frac{m}{n}}(i))q^i$. In [GORS14], the authors recover the HOMFLY polynomials of the (m, n) torus knot $T_{m,n}$ from the graded character of (the hook S_n -isotypic components of) $L_{\frac{m}{n}}$ by comparing explicit computational results of both sides:

Theorem 1.1. [GORS14, Theorem 1.1]

$$\text{HOMFLY}_{a,q}(T_{m,n}) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \text{ch}_q(\text{Hom}_{S_n}(\wedge^i(\mathfrak{h}), L_c)). \quad (1)$$

As mentioned above, the HOMFLY polynomial is the doubly graded Euler characteristic of the KhR homology. In fact, the KhR homology is a triply graded knot invariant whose full characteristic is the so-called “link superpolynomial”. As seen in Theorem 1.1, the internal q -grading of HHH corresponds to the weight space decomposition of L_c ; the Hochschild homological a -grading corresponds to the (hook-)isotypic components of $L_{\frac{m}{n}}$. The main conjecture of [GORS14] is that the third usual homological t -grading on the KhR homology should be given by a filtration on $L_{\frac{m}{n}}$:

Conjecture 1.2. [GORS14, Conjecture 1.2] *There exists a filtration on L_c whose associated t -grading in (1) yields the refined identity*

$$\text{ch}_{a,q,t}(\text{HHH}(T_{m,n})) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \text{ch}_{q,t}(\text{Hom}_{S_n}(\wedge^i(\mathfrak{h}), L_c)).$$

In *loc. cit.*, three potential candidates for this t -grading are proposed and conjectured to coincide:

- The inductive filtration F^{ind} is defined inductively using the shift functor from $H_{\frac{m}{n}}$ -modules to $H_{\frac{m}{n}+1}$ -modules and the “flipping” isomorphism $L_{\frac{m}{n}}^{S_n} \cong L_{\frac{m}{m}}$.
- The algebraic filtration F^{alg} is defined by reshuffling weight spaces under the Euler field and orthogonal complements of powers of the ideal \mathfrak{a} generated by non-constant symmetric polynomials under the Dunkl form.
- The geometric filtration F^{geom} is defined as the perverse filtration on the cohomology of a Hitchin fiber isomorphic to the compactified Jacobian of the planar singular curve $y^m = x^n$.

1.2 The inductive filtration and the Hilbert scheme of points in \mathbb{C}^2

The inductive filtration F^{ind} originated from the concept of a “good filtration” in [GS05], where the authors establish S_n -equivariant bigraded isomorphisms

$$(\mathbb{L}_{k+\frac{1}{n}})^{S_n} \cong \Gamma(\text{Hilb}_0^n(\mathbb{C}^2), \mathcal{O}(k)) \quad (2)$$

$$\mathbb{L}_{k+\frac{1}{n}} \cong \text{sign} \otimes \Gamma(\text{Hilb}_0^n(\mathbb{C}^2), \mathcal{P} \otimes \mathcal{O}(k-1)) \quad (3)$$

confirming conjectures in [BEG03b]. Here $\text{Hilb}_0^n(\mathbb{C}^2)$ is the punctual Hilbert scheme of n points on \mathbb{C}^2 and $\mathcal{O}(k)$, resp. \mathcal{P} denotes the degree- k tautological line bundle, resp. Procesi bundle of rank $n!$ on $\text{Hilb}_0^n(\mathbb{C}^2)$. The gradings on the right hand side of (2) and (3) are induced by the action of $(\mathbb{C}^*)^2$ on \mathbb{C}^2 by scaling. In this setting, only the shift functor is needed to define the inductive filtration on $\mathbb{L}_{k+\frac{1}{n}}$. As proved in [Hai98], the q, t -character of the right hand side of (2) equals to the q, t -Catalan number:

$$\text{ch}_{q,t}((\mathbb{L}_{k+\frac{1}{n}})^{S_n}) = \text{ch}_{q,t}(\Gamma(\text{Hilb}_0^n(\mathbb{C}^2), \mathcal{O}(k))) = C_{nk+1,n}(q, t). \quad (4)$$

On the other hand, as a consequence of the rational Shuffle conjecture proved in [Mel21], one also has

$$\text{ch}_{a=0,q,t}(\text{HHH}(T_{nk+1,n})) = C_{nk+1,n}(q, t).$$

These two equalities provide evidence of Conjecture 1.2 using the inductive filtration.

1.3 Algebraic and geometric filtrations and the compactified Jacobian

Let $C_{\frac{m}{n}}$ denote the closure of the plane curve $y^m = x^n$ in \mathbb{P}^2 and $J_{\frac{m}{n}}$ be the compactified Jacobian of $C_{\frac{m}{n}}$, which parametrizes rank one torsion-free coherent sheaves on $C_{\frac{m}{n}}$ of a fixed degree. On the \mathbb{C}^* -equivariant cohomology of $J_{\frac{m}{n}}$ one can define the perverse filtration P (see [OY16, 8.5.1]). Let $H_{\epsilon=1}^*(J_{\frac{m}{n}})$ denote the specialization of $H_{\mathbb{C}^*}^*(J_{\frac{m}{n}})$ at 1. It is proved in *loc. cit.* using the global Springer theory and proved in [GK23] using the BFN Springer theory that there are natural actions of the spherical Cherednik algebra $eH_{\frac{m}{n}}$ on the associated graded $\text{gr}^P H_{\epsilon=1}^*(J_{\frac{m}{n}})$ so that $\text{gr}^P H_{\epsilon=1}^*(J_{\frac{m}{n}}) \cong (\mathbb{L}_{\frac{m}{n}})^{S_n}$ as $eH_{\frac{m}{n}}$ -representations.

On the other hand, consider the arc space $\mathcal{M}_{\frac{m}{n}}$, defined as the moduli space of maps $\phi: \mathbb{P}^1 \rightarrow C_{\frac{m}{n}}$ of degree 1 such that $\phi(\infty) = \infty_C$ and $\deg(\phi^*(f) - f) \leq k-2$ for all $f \in \mathbb{C}[t^m, t^n]$ of degree k . Here ϕ^* is the induced homomorphism $\mathbb{C}[C_{\frac{m}{n}} - \infty_C] \rightarrow \mathbb{C}[\mathbb{P}^1 - \infty]$. Then according to [FGvS99, section G], $\mathbb{C}[\mathcal{M}_{\frac{m}{n}}]$ is isomorphic to $(\mathbb{L}_{\frac{m}{n}})^{S_n}$ as rings and is filtered by powers of its maximal ideal $\mathfrak{m} = \mathfrak{a}^{S_n}$.

Conjecture 1.3. [OY17, Conjecture 1.1.7] For all $i, j \in \mathbb{Z}$, $\text{gr}_j^P H^{2i}(J_{\frac{m}{n}}) = \text{gr}_{\mathfrak{m}}^{j-i} \mathbb{C}[\mathcal{M}_{\frac{m}{n}}](j)$.

In the case of $m = nk + 1$ for $k \geq 0$, the main result of [GM13] implies that

$$\sum_{i,j} \dim(\text{gr}_j^P H^{2i}(J_{k+\frac{1}{n}})) q^j t^{2i} = C_{nk+1,n}(q, qt^2). \quad (5)$$

Comparison between (5) and (4) provides a numerical correlation between $(H^*(J_{k+\frac{1}{n}}), P)$ and $((\mathbb{L}_{k+\frac{1}{n}})^{S_n}, F^{\text{ind}})$.

In this paper, we relate F^{ind} and F^{alg} .

Conjecture 1.4. [GORS14, Conjecture 4.12] On $\mathbb{L}_{\frac{m}{n}}$ when $m > n$ for coprime m, n the algebraic filtration coincides with the inductive filtration.

Theorem 1.5. (Theorem 3.16) Conjecture 1.4 holds.

Because of its definition (see Section 3.2) one can only define F^{ind} on $\mathbb{L}_{\frac{m}{n}}$ when $m > n$. But F^{ind} is defined on $(\mathbb{L}_{\frac{m}{n}})^{S_n}$ for all positive integer m coprime to n , in which cases it follows from Theorem 1.5 that F^{ind} coincides with F^{alg} .

As suggested by the discussion above (and explained in [OY17, Proposition 6.2.3]), we obtain the following as a corollary.

Theorem 1.6. When $m = nk + 1$ for $k \geq 0$, Conjecture 1.3 is true.

The method we use to prove Theorem 1.5 is induction on the powers of \mathfrak{a} . It is proved in [DO03] that $L_{\frac{m}{n}}$ can be defined as the maximal quotient of $\mathbb{C}[\mathfrak{h}]$ such that the Dunkl form is nondegenerate. The main theme of this paper is to show that the obstruction of doing induction exactly lies in the kernel of the Dunkl form $I_{\frac{m}{n}}$. We prove this statement by inspecting $I_{\frac{m}{n}}$ using the residue description provided in [Dun98, CE03, Gor13]. By decomposing elements in L_c into sums of products of symmetric polynomials and “ $\frac{m}{n}$ -harmonic polynomials” (Section 2.2), we transform the question into a linear algebra problem on dimension counts of subspaces in the direct sums of copies of coinvariant algebras (Section 4.1, 4.2). The solution to this linear algebra problem is derived from the appearance of certain distributive lattices (Section 4.3). In the case of $m = 2n - 1$, the cohomology of the Springer fiber at a minimal nilpotent element comes into the picture (Section 4.3.1). The proof is highly combinatorial and it would be interesting to geometrize it.

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2 Representations of the rational Cherednik algebra

2.1 Definitions

Definition 2.1. We define the rational Cherednik algebra \overline{H}_c associated to the Cartan $\overline{\mathfrak{h}} \subset \mathfrak{gl}_n$, Weyl group $W = S_n$ and parameter $c \in \mathbb{C}$ to be the \mathbb{C} -algebra generated by $\overline{\mathfrak{h}}, \overline{\mathfrak{h}}^*$ and W with relations

$$\begin{aligned} [x, x'] = [y, y'] = 0, \quad wxw^{-1} = w(x), \quad wyw^{-1} = w(y) \\ [y, x] = x(y) - \sum_{s \in S} c \langle \alpha_s, x \rangle \langle \alpha_s^\vee, y \rangle s \end{aligned}$$

where $x, x' \in \overline{\mathfrak{h}}, y, y' \in \overline{\mathfrak{h}}^*, w \in W, S \subset W$ is the set of simple reflections and α_s , resp. α_s^\vee , is the root, resp. coroot, associated to s .

Let $\mathfrak{h} \subset \overline{\mathfrak{h}}$ be a Cartan of \mathfrak{sl}_n . The rational Cherednik algebra H_c associated to \mathfrak{h} with parameter c can be defined similarly.

Let $\overline{\mathfrak{h}}_{\text{reg}} = \overline{\mathfrak{h}} \setminus \cup_{s \in S} \{\alpha_s = 0\}$ be the regular part of $\overline{\mathfrak{h}}$ and $\mathcal{D}(\overline{\mathfrak{h}}_{\text{reg}})$ be the ring of differential operators on $\overline{\mathfrak{h}}_{\text{reg}}$. We take x_1, \dots, x_n to be the standard basis of $\overline{\mathfrak{h}}^*$. Define the Dunkl operators

$$y_i^{(c)} := \frac{\partial}{\partial x_i} - c \sum_{s \in S} \frac{\langle \alpha_s, x_i \rangle}{\alpha_s} (1 - s).$$

(Below we may drop the superscript ^(c) when there is no ambiguity.) \overline{H}_c can be realized as a subalgebra of $\mathcal{D}(\overline{\mathfrak{h}}_{\text{reg}}) \rtimes W$ generated by x_i, y_i for $1 \leq i \leq n$ and W .

Similarly, H_c is generated by $x_1, \dots, x_n \bmod (x_1 + \dots + x_n), y_1 - y_2, \dots, y_{n-1} - y_n$ and W .

Take the polynomial representation of H_c given by $\mathbb{C}[\mathfrak{h}] \cong H_c \otimes_{S(\mathfrak{h}) \rtimes W} \mathbb{C}$. When $c = \frac{m}{n} > 0$, with m, n coprime, this representation has a finite-dimensional irreducible quotient, usually denoted by L_c .

Theorem 2.2. ([BEG03b, Theorem 1.2]) When $c = \frac{m}{n}$ for positive integer m coprime to n , the only irreducible finite-dimensional representation of H_c is L_c . Moreover, only when $c = \frac{m}{n}$ for integer m coprime to n does H_c have finite-dimensional representations.

On \overline{H}_c we have the Fourier transform defined by

$$\Phi_c(x_i) = y_i, \quad \Phi_c(y_i) = -x_i, \quad \Phi_c(w) = w \tag{6}$$

which defines the Dunkl bilinear form

$$(-, -)_c : \mathbb{C}[\mathfrak{h}] \times \mathbb{C}[\mathfrak{h}] \rightarrow \mathbb{C}, \quad (f, g)_c = [\Phi_c(f)g]|_{x_i=0}.$$

From the definition, we see that for any $\phi, \psi \in \mathbb{C}[\mathfrak{h}]$,

$$((x_i - x_j)\phi, \psi)_c = (\phi, (y_i - y_j)\psi)_c. \quad (7)$$

On the other hand, since $wx_iw^{-1} = x_{w(i)}$ and $wy_iw^{-1} = y_{w(i)}$ for any $w \in S_n$, we also have

$$(w(\phi), w(\psi))_c = (\phi, \psi)_c. \quad (8)$$

For generic c including 0, $(-, -)_c$ is non-degenerate. However, when $c = \frac{m}{n} > 0$, with m, n coprime, $(-, -)_c$ has a nonzero kernel I_c and the resulting quotient $\mathbb{C}[\mathfrak{h}]/I_c$ is exactly isomorphic to L_c ([DO03, Proposition 2.34]).

Example 2.3. When $c = \frac{k}{2}$, let $s = (12)$. Then $y_1 - y_2 = \frac{\partial}{\partial x_i} - 2c\frac{1-s}{x_1-x_2}$ and $(y_1 - y_2)((x_1 - x_2)^\ell) = 2\ell c(x_1 - x_2)^{\ell-1}$. As a result $I_{\frac{k}{2}} = ((x_1 - x_2)^k)$ and $\dim L_{\frac{k}{2}} = k$.

Under the W -action, L_c decomposes into isotypic components

$$L_c = \bigoplus_{\sigma \in \text{Irrep}W} e_\sigma L_c,$$

where $e_\sigma L_c = \sigma \otimes \text{Hom}_W(\sigma, L_c)$. In particular, for $e = \frac{1}{n!} \sum_{w \in W} w$ and $e_- = \frac{1}{n!} \sum_{w \in W} (-1)^{\text{sign}(w)} w$, $eL_c = e_{\text{triv}}L_c$ while $e_-L_c = e_{\text{sign}}L_c$. For any two subspaces U, V of $\mathbb{C}[\mathfrak{h}]$ or L_c , we write $U \perp_c V$ when U and V are orthogonal with respect to $(-, -)_c$.

Lemma 2.4. $e_\sigma L_c \perp_c e_{\sigma'} L_c$ if $\sigma \neq \sigma'$.

Proof. It is a consequence of the property (8) which says that $(-, -)_c$ is W -invariant. \square

2.2 c -Harmonic polynomials

From now on we will assume c satisfies the assumption in Theorem 2.2. Inside $\mathbb{C}[\mathfrak{h}]$, take the ideal $\mathfrak{a} := (\mathbb{C}[\mathfrak{h}]_+^W)$. Let $\delta := \prod_{i < j} (x_i - x_j)$ denote the Vandermonde determinant. Define the space of c -Harmonic polynomials to be $\mathcal{H}_c := \mathbb{C}[y_1 - y_2, \dots, y_{n-1} - y_n]\delta$.

Lemma 2.5. For non-negative b_2, \dots, b_n , if $\sum_{i=2}^n b_i = \frac{n(n-1)}{2}$ and $\{b_2, \dots, b_n\} \neq \{1, 2, \dots, n-1\}$, then $y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta = 0$

Proof. For any $\sigma \in W$,

$$y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta = \sigma(y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta) = y_{\sigma(2)}^{b_2} y_{\sigma(3)}^{b_3} \dots y_{\sigma(n)}^{b_n} ((-1)^{\text{sign}(\sigma)} \delta)$$

because for degree reasons $y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta$ is a constant. From this we deduce

$$\sum_{\sigma \in W} (-1)^{\text{sign}(\sigma)} \sigma(y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta) = n! y_2^{b_2} y_3^{b_3} \dots y_n^{b_n} \delta. \quad (9)$$

On the other hand, the polynomial $\sum_{\sigma} (-1)^{\text{sign}(\sigma)} \sigma(x_2^{b_2} x_3^{b_3} \dots x_n^{b_n})$ is skew-symmetric of degree the same as δ and hence has to equal to $\lambda \delta$ for some constant λ because of

$$\mathbb{C}[\mathfrak{h}]^{\text{sign}} = \delta \mathbb{C}[\mathfrak{h}]. \quad (10)$$

However, δ expands into

$$\sum_{\sigma \in W} (-1)^{\text{sign}(\sigma)} x_{\sigma(2)} x_{\sigma(3)}^2 \dots x_{\sigma(n)}^{n-1} \quad (11)$$

since $\{b_2, \dots, b_n\} \neq \{1, 2, \dots, n-1\}$, by comparing the degrees of terms in δ and $\sum_{\sigma} (-1)^{\text{sign}(\sigma)} \sigma(x_2^{b_2} x_3^{b_3} \dots x_n^{b_n})$, we conclude that $\lambda = 0$ and the lemma follows. \square

Similar to the classical result $\mathbb{C}[\mathfrak{h}] = \mathfrak{a} \oplus^{\perp_0} \mathcal{H}_0$ ([CG10, 6.3]), there is the following analogue:

Proposition 2.6. *When $c = \frac{m}{n} > 1$ with $(m, n) = 1$, we have that $\mathbb{C}[\mathfrak{h}] = \mathfrak{a} \oplus \mathcal{H}_c$ with $\mathfrak{a} \perp_c \mathcal{H}_c$.*

Proof. Since the image of δ under the action of any symmetric polynomial in y_i 's is still scew-symmetric, because of (10) the image has to be 0 and so $\mathcal{H}_c \perp_c \mathfrak{a}$. On the other hand, since $\mathbb{C}[\mathfrak{h}]/\mathfrak{a} \cong \mathbb{C}[W]$, it is sufficient to prove that $\dim \mathcal{H}_c = n!$.

We first claim that $y_2 y_3^2 \cdots y_n^{n-1} \delta \neq 0$.

When $c > 1$ the image of δ is nonzero along the projection $\mathbb{C}[\mathfrak{h}] \rightarrow L_c$ as ([BEG03b, 4.3])

$$\delta eL_c = e_- L_c \cong eL_{c-1} \neq 0.$$

This implies that there is a polynomial ϕ such that $(\delta, \phi)_c \neq 0$. But by Lemma 2.4, we may assume $\phi \in e_- L_c = \delta eL_c$. For degree reasons, we must have that ϕ is a nonzero multiple of δ . As a result, we have $(\delta, \delta)_c \neq 0$, which by (11) expands to

$$\left(\sum_{\sigma \in W} (-1)^{\text{sign}(\sigma)} y_{\sigma(2)} y_{\sigma(3)}^2 \cdots y_{\sigma(n)}^{n-1} \right) \delta. \quad (12)$$

By (9), (12) equals to $n! y_2 y_3^2 \cdots y_n^{n-1} \delta \neq 0$ and the claim follows.

It remains to show that the elements $y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta$, satisfying $0 \leq a_i \leq i-1$, which are nonzero from the discussion above, are linearly independent.

Suppose otherwise, i.e.

$$\sum_{\sum a_i = N} \alpha_{a_2, \dots, a_n} y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta = 0,$$

where $0 \leq a_i \leq i-1$ and $N \in \mathbb{Z}_{\geq 0}$. Let (A_2, \dots, A_n) be the maximal element inside the set $\{(a_1, \dots, a_{n-1}) \mid \sum a_i = N, 0 \leq a_i \leq i-1\}$ under the lexicographical order¹

Then

$$y_2^{1-A_2} y_3^{2-A_3} \cdots y_n^{n-1-A_n} \sum_{\sum a_i = N} \alpha_{a_2, \dots, a_n} y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta = 0.$$

As a consequence of Lemma 2.5

$$0 = y_2^{1-A_2} y_3^{2-A_3} \cdots y_n^{n-1-A_n} \sum_{\sum a_i = N} \alpha_{a_2, \dots, a_n} y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta = \alpha_{a_2, \dots, a_n} y_2 y_3^2 \cdots y_n^{n-1} \delta,$$

which was proved to be nonzero above. This yields a contradiction. Therefore, the elements $y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta$ satisfying $0 \leq a_i \leq i-1$ are linearly independent and the lemma holds. \square

In view of the proof, we actually have that Proposition 2.6 holds whenever $\delta \notin I_c$.

Lemma 2.7. *The polynomials $\phi_{a_2, \dots, a_n} := x_2^{a_2} x_3^{a_3} \cdots x_n^{a_n}$, satisfying $0 \leq a_i \leq i-1$ form a basis of $\mathbb{C}[\mathfrak{h}]/\mathfrak{a}$. Moreover, any $\phi \in \mathbb{C}[\mathfrak{h}]$ can be uniquely expressed as $\phi = \sum h_i \psi_i$ with $h_i \in \mathcal{H}_c$ and $\psi_i \in \mathbb{C}[\mathfrak{h}]^W$. In other words, $\mathbb{C}[\mathfrak{h}] = \mathcal{H}_c \cdot \mathbb{C}[\mathfrak{h}]^W$.*

Proof. It follows from Lemma 2.6 and the fact that $\mathbb{C}[\mathfrak{h}]$ is the Galois extension of $\mathbb{C}[\mathfrak{h}]^W$ with respect to the polynomial $\prod_{i=1}^n (x - x_i)$ with basis given by ϕ_{a_2, \dots, a_n} , $0 \leq a_i \leq i-1$. \square

Corollary 2.8. *$I_c \cap \mathcal{H}_c = \{0\}$ when $c > 1$.*

Proof. As in the proof of Lemma 2.6, $(y_2^{1-a_2} y_3^{2-a_3} \cdots y_n^{n-1-a_n})(y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n}) \delta$ is a nonzero constant. Moreover, if $\phi \in I_c$, then $(y_i - y_{i+1})\phi \in I_c$ for any $1 \leq i \leq n-1$ because of (7). Since nonzero constants are not contained in I_c , we conclude that $y_2^{a_2} y_3^{a_3} \cdots y_n^{a_n} \delta \notin I_c$ for all $0 \leq a_i \leq i-1$ and the corollary follows. \square

¹i.e. for (a_1, \dots, a_{n-1}) and (a'_1, \dots, a'_{n-1}) , if for some $1 \leq i \leq n-1$, $a_1 = a'_1, \dots, a_i = a'_i$ and $a_{i+1} > a'_{i+1}$ then $(a_1, \dots, a_{n-1}) > (a'_1, \dots, a'_{n-1})$.

To avoid extra notations, below we will simply use \mathfrak{a} and \mathcal{H}_c to denote their images under $\mathbb{C}[\mathfrak{h}] \rightarrow L_c$ when there is no ambiguity. That is to say, we have a direct sum decomposition $L_c = \mathfrak{a} \oplus \mathcal{H}_c$ and $\mathcal{H}_c = \mathfrak{a}^\perp_c$, where \perp_c denotes orthogonal complement with respect to $(-, -)_c$.

There is a W -action on \mathcal{H}_c by permuting the y_i 's. Under this action, \mathcal{H}_c is isomorphic to the regular representation of W . Decompose $\mathcal{H}_c = \bigoplus_\sigma \mathcal{H}_c^\sigma$ so that $\mathcal{H}_c^\sigma = \sigma \otimes \text{Hom}_W(\sigma, \mathcal{H}_c)$.

Corollary 2.9. $e_\sigma L_c = \mathcal{H}_c^\sigma \cdot eL_c$.

Proof. Clearly $\mathcal{H}_c^\sigma \cdot eL_c \subset e_\sigma L_c$. Moreover, it follows from Lemma 2.7 that $L_c = \mathcal{H}_c \cdot eL_c = (\bigoplus_\sigma \mathcal{H}_c^\sigma) \cdot eL_c = \bigoplus_\sigma \mathcal{H}_c^\sigma \cdot eL_c$ and hence the opposite inclusion follows. \square

3 The power filtration and the inductive filtration

3.1 The algebraic filtration and the power filtration

3.1.1 The power filtration

Let $p_i := \sum_{j=1}^n x_j^i$ be the power sum symmetric polynomial of degree i . Then $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[p_1, p_2, \dots, p_n]/(p_1)$. The following three elements in H_c form an \mathfrak{sl}_2 -triple

$$\mathbf{e} = \frac{1}{2}p_2 - \frac{1}{2n}p_1^2 = \frac{1}{2n} \sum_{i < j} (x_i - x_j)^2, \quad \mathbf{f} = -\Phi_c(\mathbf{e}) = -\frac{1}{2n} \sum_{i < j} (y_i - y_j)^2,$$

$$\mathbf{h} = \frac{1}{2} \sum (x_i y_i + y_i x_i) - \frac{1}{2n} (p_1 \Phi_c(p_1) + \Phi_c(p_1) p_1) = \frac{1}{2n} \sum_{i < j} \left((x_i - x_j)(y_i - y_j) + (y_i - y_j)(x_i - x_j) \right)$$

$$[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}, \quad [\mathbf{e}, \mathbf{f}] = \mathbf{h}$$

Lemma 3.1. On L_c , \mathbf{h} acts by $\frac{1}{n} \sum_{i < j} (x_i - x_j)(y_i - y_j) - \mu$ where $\mu = \frac{(m-1)(n-1)}{2}$.

Remark 3.2. μ is the Milnor number of the singular curve $\{x^m = y^n\}$.

Proof. We compute that

$$\begin{aligned} \mathbf{h} &= \frac{1}{2n} \sum_{i < j} \left((x_i - x_j)(y_i - y_j) - (x_i - x_j)(y_i - y_j) \right) \\ &= \frac{1}{2n} \sum_{i < j} \left(2(x_i - x_j)(y_i - y_j) + 2|S| - \sum_{i < j} \sum_{s \in S} c(\alpha_s, x_i - x_j) \langle y_i - y_j, \alpha_s^\vee \rangle s \right) \\ &= \frac{1}{2n} \sum_{i < j} \left(2(x_i - x_j)(y_i - y_j) + \left(2|S| - c \left(\binom{n}{2} + \binom{n-2}{2} - 3 \right) \sum_{s \in S} s \right) \right). \end{aligned}$$

Here S is the set consisting of all reflections with $|S| = \binom{n}{2}$ and $\binom{n}{2} - \binom{n-2}{2} = 2n - 3$ is the number of α_s such that $\langle \alpha_s, x_i - x_j \rangle \neq 0$. Moreover, $\langle \alpha_s, x_i - x_j \rangle$ equals to 2 when $\alpha_s = x_i - x_j$ and equals to ± 1 when $s = (ik)$ or (jk) for some $k \neq i, j$. Therefore

$$\mathbf{h} = \frac{1}{2n} \sum_{i < j} \left(2(x_i - x_j)(y_i - y_j) + (2|S| - 2nc \sum_{s \in S} s) \right).$$

Since $\sum_{s \in S} s$ is in the center of the group algebra of W , it acts by a scalar on L_c , which is $|S|$. So finally

$$\mathbf{h} = \frac{1}{n} \sum_{i < j} \left((x_i - x_j)(y_i - y_j) + (1 - nc)|S| \right) = \frac{1}{n} \sum_{i < j} \left((y_i - y_j)(x_i - x_j) - (1 - nc)|S| \right).$$

We conclude the lemma by

$$\frac{1}{n}(nc - 1)|S| = \frac{(m-1)(n-1)}{2} = \mu. \quad \square$$

The left action of \mathfrak{h} gives a decomposition of L_c into weight spaces $\bigoplus_k L_c(k)$ where $L_c(k) = \{v \in L_c | \mathfrak{h}v = kv\}$. As a corollary of Lemma 3.1, $\mathfrak{h} \cdot 1 = -\mu$. Therefore 1 is a eigenvector vector of minimal lowest weight $-\mu$ and \mathbf{e}^μ is a eigenvector of maximal highest weight μ .

Note that $[\mathbf{e} - \mathbf{f}, x] = [-\mathbf{f}, x] = y$, and $[\mathbf{e} - \mathbf{f}, y] = -x$. Also, because the action of $\langle \text{ad}(\mathbf{e}), \text{ad}(\mathbf{f}), \text{ad}(\mathbf{h}) \rangle$ on H_c is locally finite and integrable, [BEG03a, Remark after 3.8] tells us that the Fourier transform (6) can also be expressed by

$$\Phi_c = \text{Ad}(e^{\frac{i\pi}{2}(\mathbf{e}-\mathbf{f})}) : H_c \rightarrow H_c. \quad (13)$$

This allows us to define

$$\bar{\Phi}_c : L_c \rightarrow L_c, \quad \phi \mapsto e^{\frac{i\pi}{2}(\mathbf{e}-\mathbf{f})}\phi \quad (14)$$

where $e^{\frac{i\pi}{2}(\mathbf{e}-\mathbf{f})}$ acts by left action. The lemma below gives a concrete characterization of $\bar{\Phi}_c$.

Lemma 3.3. *Suppose $0 \neq \phi \in L_c(k)$. If $k < 0$, then $\bar{\Phi}_c(\phi)$ is a nonzero multiple of $\mathbf{e}^{-k}\phi$; if $k > 0$, then $\bar{\Phi}_c(\phi)$ is a nonzero multiple of $\mathbf{f}^k\phi$. In particular, we have that $\bar{\Phi}_c(1)$ is a nonzero multiple of \mathbf{e}^μ .*

Proof. Since $\Phi_c(\mathbf{e}) = -\mathbf{f}$ and $\Phi_c(\mathbf{f}) = -\mathbf{e}$, we know that $\bar{\Phi}_c$ preserves $\langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle$ -subrepresentations of L_c . On the other hand, because $\Phi_c(\mathbf{h}) = -\mathbf{h}$, for any $\phi \in L_c$ we have $\mathfrak{h}\bar{\Phi}_c(\phi) = -\bar{\Phi}_c\mathfrak{h}(\phi)$. As a result $\bar{\Phi}_c(L_c(k)) = L_c(-k)$. Finally, notice that when $\phi \in L_c(k)$ with $k < 0$, it follows that $0 \neq \mathbf{e}^{-k}\phi \in L_c(-k) \cap \langle \mathbf{e}, \mathbf{f}, \mathbf{h} \rangle\phi$, which is one-dimensional. Hence $\mathbf{e}^{-k}\phi = C\bar{\Phi}_c(\phi)$ for some nonzero constant C . The argument is similar for the case when $k > 0$. \square

The algebraic filtration $F^{\text{alg}'}$ in [GORS14, Definition 4.6] is defined by

$$F_i^{\text{alg}'}(L_c) = \left(\sum_{2j-k > i} (\mathfrak{a}^j)(k) \right)^{\perp c}.$$

Below we use the Fourier transform to reconstruct $F^{\text{alg}'}$.

Definition 3.4. *For all $c > 0$, using (14) the power filtration on L_c is defined by*

$$F_i^{\mathfrak{a}}(L_c) = \bar{\Phi}_c[(\mathfrak{a}^{i+1})^{\perp c}] = \Phi_c[(\mathfrak{a}^{i+1})^{\perp c}]\mathbf{e}^\mu.$$

We let $\mathbb{C}[\mathfrak{h}]_j^W$ be the sub vector space spanned by $p_{i_1}p_{i_2}\cdots p_{i_j}$ for $2 \leq i_1, \dots, i_j \leq n$ and $\mathbb{C}[\mathfrak{h}]_{\leq j}^W = \bigoplus_{\ell=0}^j \mathbb{C}[\mathfrak{h}]_\ell^W$.

Lemma 3.5. *$\phi \in (\mathfrak{a}^i)^{\perp c} \subset L_c$ if and only if for all $\psi \in \mathbb{C}[\mathfrak{h}]_i^W$, $\Phi_c(\psi)\phi = 0$.*

Proof. “ \Leftarrow ” is clear. To show “ \Rightarrow ”, assume $\phi \in (\mathfrak{a}^i)^{\perp c}$ but $\Phi_c(\psi)\phi \neq 0$ for some $\psi \in \mathbb{C}[\mathfrak{h}]_i^W$. Then since $(-, -)_c$ is non-degenerate on L_c , there exists some $\xi \in L_c$ so that $(\xi, \Phi_c(\psi)\phi)_c = (\xi\psi, \phi)_c \neq 0$ is a nonzero constant. This contradicts the assumption that $\phi \in (\mathfrak{a}^i)^{\perp c}$ as $\xi\psi \in \mathfrak{a}^i$. \square

Lemma 3.6. *For $\phi \in [(\mathfrak{a}^{i+1})^{\perp c}]^W$, $(y_i - y_{i+1})\phi \in (\mathfrak{a}^i)^{\perp c}$ for all i .*

Proof. Take $\psi \in \mathbb{C}[\mathfrak{h}]_i^W$. Then $\Phi_c(\psi)\phi$ is symmetric and also lies in $\mathfrak{a}^{\perp c} = \mathcal{H}_c$. Therefore $\Phi_c(\psi)\phi$ is a constant and $\Phi_c(\psi)(y_i - y_{i+1})\phi = 0$. Now apply the last lemma. \square

3.1.2 The Kazhdan filtration

Definition 3.7. [Gin18, 3.2] *The Kazhdan filtration K^F on a vector space V associated to an ascending \mathbb{Z} -filtration F and a \mathbb{Z} -grading $V = \bigoplus_{k \in \mathbb{Z}} V(k)$ is*

$$K_i^F(V) = \sum_{2j+k \leq i} F_j(V)(k).$$

Definition 3.8. Define the algebraic filtration F^{alg} on L_c to be the Kazhdan filtration associated to $F^{\mathfrak{a}}$ and the \mathfrak{h} -grading, i.e.

$$F_i^{\text{alg}} = \sum_{2j+k \leq i} \overline{\Phi}_c((\mathfrak{a}^{j+1})^{\perp c})(k).$$

Lemma 3.9. $F^{\text{alg}} = F^{\text{alg}'}$.

Proof. For fixed i and k we have

$$F_i^{\text{alg}}(k) = \sum_{2j+k \leq i} \overline{\Phi}_c((\mathfrak{a}^{j+1})^{\perp c})(k) = \overline{\Phi}_c((\mathfrak{a}^{\lceil \frac{i-k+1}{2} \rceil})^{\perp c})(k),$$

$$F_i^{\text{alg}'}(k) = \left(\sum_{k < 2j-i} (\mathfrak{a}^j)(k) \right)^{\perp c} = (\mathfrak{a}^{\lceil \frac{i+k+1}{2} \rceil})^{\perp c}(k).$$

Take $P \in \overline{\Phi}_c[(\mathfrak{a}^{\lceil \frac{i-k+1}{2} \rceil})^{\perp c}]$ a polynomial in $y_1 - y_2, \dots, y_{n-1} - y_n$ of degree $\mu - k$. Take $\phi \in \mathfrak{a}^{\lceil \frac{i+k+1}{2} \rceil}$ of degree $\mu + k$ so that $\phi \in L_c(k)$. Then

$$(\phi, P\mathbf{e}^\mu)_c = (\overline{\Phi}_c(P), \overline{\Phi}_c(\phi)\mathbf{e}^\mu)_c.$$

As a result $F_i^{\text{alg}}(k) = F_i^{\text{alg}'}(k)$ if and only if $\overline{\Phi}_c(\mathfrak{a}^{\lceil \frac{i+k+1}{2} \rceil})(-k) = \mathfrak{a}^{\lceil \frac{i-k+1}{2} \rceil}(-k)$. After change of variables, this latter condition is equivalent to $\overline{\Phi}_c(\mathfrak{a}^u)(v) = \mathfrak{a}^{u+v}(v)$.

Now by Lemma 3.3, $\overline{\Phi}_c(\mathfrak{a}^u)(v) = \overline{\Phi}_c(\mathfrak{a}^u(-v))$, which equals to $\mathbf{e}^v(\mathfrak{a}^u(-v)) = (\mathbf{e}^v \mathfrak{a}^u)(v)$ when $v > 0$ and equals to $\mathbf{f}^{-v}(\mathfrak{a}^u(-v)) = (\mathbf{f}^{-v} \mathfrak{a}^u)(v)$ when $v < 0$. Since $(\mathbf{e}^v \mathfrak{a}^u)(v)$ and $(\mathbf{f}^{-v} \mathfrak{a}^u)(v)$ both lie in $\mathfrak{a}^{u+v}(v)$, we conclude

$$\overline{\Phi}_c(\mathfrak{a}^u)(v) \subset \mathfrak{a}^{u+v}(k),$$

from which we also have

$$\overline{\Phi}_c(\mathfrak{a}^{u+v}(v)) = (\overline{\Phi}_c(\mathfrak{a}^{u+v}))(-v) \subset \mathfrak{a}^u(-v).$$

Now applying $\overline{\Phi}_c$ gives us $\mathfrak{a}^{u+v}(v) \subset \overline{\Phi}_c(\mathfrak{a}^u(-v)) = \overline{\Phi}_c(\mathfrak{a}^u)(v)$. Therefore $\overline{\Phi}_c(\mathfrak{a}^u)(v) = \mathfrak{a}^{u+v}(v)$ and $F^{\text{alg}} = F^{\text{alg}'}$. \square

3.2 The inductive filtration

We have the following isomorphisms

$$\text{when } m, n > 1, \quad \mathfrak{e}L_{\frac{m}{n}} \cong \mathfrak{e}L_{\frac{n}{m}} \quad ([\text{CEE09}, 8.2]) \quad (15)$$

$$\text{when } c > 1, \quad L_c \cong H_c \mathfrak{e}_- \otimes_{\mathfrak{e}H_{c-1} \mathfrak{e}} \mathfrak{e}L_{c-1} \quad ([\text{GS05}, \text{Theorem 1.6}]) \quad (16)$$

To view $H_c \mathfrak{e}_-$ as a right $\mathfrak{e}H_{c-1} \mathfrak{e}$ -module, one uses the identification $\mathfrak{e}_- H_c \mathfrak{e}_- \cong \delta \mathfrak{e}H_{c-1} \mathfrak{e} \delta^{-1}$ ([BEG03b, Proposition 4.1]). The isomorphism (16) implies an embedding $\mathfrak{e}L_{c-1} \hookrightarrow \mathfrak{e}L_c$: $em \mapsto \mathfrak{e}_- \otimes em$ and an isomorphism

$$\mathfrak{e}_- L_c \cong \delta \mathfrak{e}L_{c-1} \quad (17)$$

when $c > 1$. Using (15) and (16), we can define two inductive filtrations on L_c when $c > 1$ and on $\mathfrak{e}L_c$ when $c > 0$, denoted by F^{ind} and $F^{\text{ind}'}$ respectively.

First we give a partial order on the positive rational numbers in the following way: for coprime pairs (m, n)

$$\frac{m}{n} \prec \frac{m+n}{n}; \text{ if } n < m, \text{ then } \frac{m}{n} \prec \frac{m}{n}. \quad (18)$$

We can then use the Euclidean algorithm to go from $c = \frac{m}{n}$ to $\frac{1}{n'}$ for some integer $n' > 1$ through a chain of rational numbers decreasing under the order $(\mathbb{Q}_{>0}, \prec)$. For example, if $c = \frac{13}{5}$, then we have $\frac{13}{5} \succ \frac{3}{5} \succ \frac{5}{3} \succ \frac{2}{3} \succ \frac{3}{2} \succ \frac{1}{2}$.

Definition 3.10. [GORS14, Theorem 4.1] For the base case when $m = 1$, F^{ind} and $F^{\text{ind}'}$ are defined trivially on L_c :

$$0 = F_{-1}^* eL_{\frac{1}{n}} \subset F_0^* eL_{\frac{1}{n}} = eL_{\frac{1}{n}} = L_{\frac{1}{n}}.$$

where $\star = \text{ind}$ or ind' . Next F^{ind} and $F^{\text{ind}'}$ are defined inductively under the order $(\mathbb{Q}_{>0}, <)$ using the isomorphisms (15) and (16).

- When the filtration on the right hand side of (16) is the tensor product filtration where H_c is endowed with the order filtration F^{ord} such that $\deg y = 1$ and $\deg x = \deg w = 0$ we get F^{ind} .
- When the filtration on the right hand side of (16) is the tensor product filtration where H_c is endowed with the Bernstein filtration F^{Bern} such that $\deg y = \deg x = 1$ and $\deg w = 0$ we obtain $F^{\text{ind}'}$.

Lemma 3.11. $F^{\text{ind}'}$ is the Kazhdan filtration associated to F^{ind} and the \mathbf{h} -grading.

Proof. It is sufficient to show that the Bernstein filtration F^{Bern} on $\mathcal{D}(\bar{\mathfrak{h}}_{\text{reg}})$ (containing H_c) defined by $\deg(x_i) = \deg(\partial_{x_i}) = 1$ is the Kazhdan filtration associated to the the \mathbf{h} -grading and the order filtration F^{ord} defined by $\deg(x_i) = 0$ and $\deg(\partial_{x_i}) = 1$.

By definition, $F_i^{\text{ord}}(\mathcal{D}(\bar{\mathfrak{h}}_{\text{reg}})) = \sum_{|\alpha| \leq i} \mathcal{O}_{\bar{\mathfrak{h}}_{\text{reg}}}^{\alpha} \partial_x^{\alpha}$ with $|\alpha| = \sum_i \alpha_i$ whose associated Kazhdan filtration is

$$K_i^{\text{ord}}(\mathcal{D}(\bar{\mathfrak{h}}_{\text{reg}})) = \sum_{2j+k \leq i} \sum_{|\alpha| \leq j} \mathcal{O}_{\bar{\mathfrak{h}}_{\text{reg}}}^{\alpha} \partial_x^{\alpha}(k).$$

Now note that the \mathbf{h} -grading of x_i is 1 while that of ∂_{x_i} is -1 . Therefore

$$K_i^{\text{ord}}(\mathcal{D}(\bar{\mathfrak{h}}_{\text{reg}})) = \sum_{2j+|\beta|-|\alpha| \leq i} \sum_{|\alpha| \leq j} \xi_{\alpha, \beta} x^{\beta} \partial_x^{\alpha} = \sum_{|\beta|+|\alpha| \leq i} \xi_{\alpha, \beta} x^{\beta} \partial_x^{\alpha}.$$

where $\xi_{\alpha, \beta} \in \mathbb{C}$. This is exactly F^{Bern} and the lemma is proved. \square

Corollary 3.12. The equality $F^{\text{alg}} = F^{\text{ind}'}$ is equivalent to the equality $F^{\mathbf{a}} = F^{\text{ind}}$.

Lemma 3.13. (1) $\mathbf{f} \cdot F_i^{\mathbf{a}} L_c \subset F_{i+1}^{\mathbf{a}} L_c$, for $c > 0$

(2) $\mathbf{f} \cdot F_i^{\text{ind}} eL_c \subset F_{i+1}^{\text{ind}} eL_c$ for $c > 0$ and $\mathbf{f} \cdot F_i^{\text{ind}} L_c \subset F_{i+1}^{\text{ind}} L_c$ for $c > 1$.

(3) $\mathbf{e} \cdot F_i^{\mathbf{a}} L_c \subset F_{i-1}^{\mathbf{a}} L_c$, for $c > 0$

(4) $\mathbf{e} \cdot F_i^{\text{ind}} eL_c \subset F_{i-1}^{\text{ind}} eL_c$ for $c > 0$ and $\mathbf{e} \cdot F_i^{\text{ind}} L_c \subset F_{i-1}^{\text{ind}} L_c$ for $c > 1$.

(5) Assume v is a highest weight vector with respect to the \mathfrak{sl}_2 triple $\{\mathbf{e}, \mathbf{f}, \mathbf{h}\}$ and $v \in F_j^{\text{ind}} L_c \setminus F_{j-1}^{\text{ind}} L_c$. Then for $i \geq 0$ such that $\mathbf{f}^i v \neq 0$, we have $\mathbf{f}^i v \in F_{j+i}^{\text{ind}} L_c \setminus F_{j+i-1}^{\text{ind}} L_c$. The same holds for $F^{\mathbf{a}}$ in place F^{ind} .

Proof. We will only prove (1) and (2). (3) and (4) can be shown by an analogous argument. (5) follows from (1)-(4).

By definition, (1) is to say $\mathbf{f} \cdot \bar{\Phi}_c((\mathbf{a}^{i+1})^{\perp c}) \subset \bar{\Phi}_c((\mathbf{a}^{i+2})^{\perp c})$, or equivalently $\mathbf{e} \cdot (\mathbf{a}^{i+1})^{\perp c} \subset (\mathbf{a}^{i+2})^{\perp c}$. By Lemma 3.5 and equation (7): $((x_i - x_j)\phi, \psi)_c = (\phi, (y_i - y_j)\psi)_c$, it suffices to show $\mathbf{f} \cdot \mathbb{C}[\mathfrak{h}]_{i+2}^W \subset \mathbb{C}[\mathfrak{h}]_{i+1}^W$.

Recall that $\mathbf{f} = -\frac{1}{2n} \sum_{i < j} (y_i - y_j)^2$. We write $\tilde{\nabla} f = (\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j})_{i < j}$ and use [Gor13, Corollary 5.3] to compute that for $2 \leq j_1, \dots, j_{i+2} \leq n$

$$\begin{aligned} -2n\mathbf{f} \cdot (p_{j_1} \cdots p_{j_{i+2}}) &= (-2n\mathbf{f} \cdot p_{j_1}) p_{j_2} \cdots p_{j_{i+2}} + \cdots + p_{j_1} p_{j_2} \cdots (-2n\mathbf{f} \cdot p_{j_{i+2}}) \\ &\quad + (\tilde{\nabla} p_{j_1} \cdot \tilde{\nabla} p_{j_2}) \cdots p_{j_{i+2}} + p_{j_1} \cdots (\tilde{\nabla} p_{j_i} \cdot \tilde{\nabla} p_{j_{i+2}}) \end{aligned}$$

which does lie in $\mathbb{C}[\mathfrak{h}]_{i+1}^W$.

Next, we prove (2) by induction. Trivially, part (2) of the lemma holds for $eL_{1/n}$ for all $n > 1$. Now for $c = m/n$ where $m > 1$ we assume the statement holds for all $(m', n') \prec (m, n)$. For eL_c we may assume $c > 1$ otherwise we take $eL_{1/c}$ given axiom (1) for F^{ind} . Thus we only need to prove the statement for L_c .

Take $\sum \xi_k \otimes \eta_k \in \mathbf{H}_c \delta e_- \otimes_{e_- \mathbf{H}_c e_-} e \mathbf{L}_{c-1}$ so that $\xi_k \in F_\alpha \mathbf{H}_c \delta e_-$ and $\eta_k \in F_\beta^{\text{ind}} e \mathbf{L}_{c-1}$ and $\alpha + \beta \leq i$. Apply $\Phi_c(p_2)$ and we obtain

$$\sum \Phi_c(p_2) \xi_i \otimes \eta_i = \sum [\Phi_c(p_2), \xi_i] \otimes \eta_i + \xi_i \otimes \Phi_c(p_2) \eta_i.$$

Here $[\Phi_c(p_2), \xi_i] \in F_{\alpha+1}$ and $\Phi_c(p_2) \eta_i \in F_{\alpha+1}^{\text{ind}}$ by the inductive hypothesis. Hence (2) of the lemma follows. \square

Corollary 3.14. (1) $\mathbf{e}(F_i^{\text{alg}} \mathbf{L}_c) \subset F_i^{\text{alg}} \mathbf{L}_c$ and $\mathbf{e}(F_i^{\text{ind}'} \mathbf{L}_c) \subset F_i^{\text{ind}'} \mathbf{L}_c$.

(2) $\mathbf{f}(F_i^{\text{alg}} \mathbf{L}_c) \subset F_i^{\text{alg}} \mathbf{L}_c$ and $\mathbf{f}(F_i^{\text{ind}'} \mathbf{L}_c) \subset F_i^{\text{ind}'} \mathbf{L}_c$.

(3) $\overline{\Phi}_c(F_i^{\text{alg}} \mathbf{L}_c) = F_i^{\text{alg}} \mathbf{L}_c$ and $\overline{\Phi}_c(F_i^{\text{ind}'} \mathbf{L}_c) = F_i^{\text{ind}'} \mathbf{L}_c$.

(4) Assume v is a highest weight vector under the \mathfrak{sl}_2 triple $\{\mathbf{e}, \mathbf{f}, \mathbf{h}\}$ and $v \in F_j^{\text{ind}'} \mathbf{L}_c \setminus F_{j-1}^{\text{ind}'} \mathbf{L}_c$. Then for $i \geq 0$ such that $\mathbf{f}^i v \neq 0$, $\mathbf{f}^i v \in F_j^{\text{ind}'} \mathbf{L}_c \setminus F_{j-1}^{\text{ind}'} \mathbf{L}_c$. The same holds for F^{alg} in place $F^{\text{ind}'}$.

Proof. Notice that $\mathbf{e} \cdot \mathbf{L}_c(k) \subset \mathbf{L}_c(k+2)$ while $\mathbf{f} \cdot \mathbf{L}_c \subset \mathbf{L}_c(k-2)$. Hence (1) and (2) of the corollary follow from Lemma 3.13 and the fact that F^{alg} is the Kazhdan filtrations associated to F^α and $F^{\text{ind}'}$ is the Kazhdan filtrations associated to F^{ind} . (3) follows from (1) and (2) plus Lemma 3.3. (4) follows from (1),(2) and (3). \square

3.3 Some first relations between F^α and F^{ind}

It is discussed in [GORS14, Theorem 4.8] that F^{alg} is compatible with (15) and (17). It is not hard to see that F^α is also compatible with (15) and (17): According to [CEE09, 8.2] the isomorphism $e \mathbf{L}_{\frac{m}{n}} \cong e \mathbf{L}_{\frac{n}{m}}$ can be defined by (up to scalars)

$$p_i(x_1, \dots, x_n) \mapsto p_i(x_1, \dots, x_m),$$

which is clearly a filtered homomorphism under F^α . Also, the following lemma shows that F^α is compatible with (17):

Lemma 3.15. $\delta((\mathfrak{a}^i)^{\perp c})^W \subset (\mathfrak{a}^i)^{\perp}$ and $\Phi_c(\delta)((\mathfrak{a}^i)^{\perp})^{\text{sign}} \subset (\mathfrak{a}^i)^{\perp}$.

Proof. We will only show that $\delta((\mathfrak{a}^i)^{\perp})^W \subset (\mathfrak{a}^i)^{\perp}$. The proof of the other statement is similar. Assume there exists $\phi \in \mathbb{C}[\mathfrak{h}]_i^W$ and $\psi \in ((\mathfrak{a}^i)^{\perp})^W$ such that $\Phi_c(\phi)(\delta\psi) \neq 0$. Then since $\Phi_c(\phi)(\delta\psi) \in \mathbb{C}[\mathfrak{h}]^{\text{sign}} = \delta \mathbb{C}[\mathfrak{h}]^W$, which intersects $\mathbf{H}_c = \mathfrak{a}^{\perp}$ only at $\{0\}$, there exists some symmetric polynomial η (possibly constant) satisfying $\Phi_c(\eta)\Phi_c(\phi)(\delta\psi) = \delta$. From the proof of Lemma 2.6, we see that $\Phi_c(\delta)\delta \neq 0$. As a result, $\Phi_c(\delta\eta\phi)(\delta\psi)$ is a nonzero constant. As a result,

$$0 \neq (\Phi_c(\delta)(\delta\psi), \eta\phi)_c = (\delta\psi, \delta\eta\phi)_c = (\delta, \delta)_c(\psi, \delta\eta)_{c-1} \quad (19)$$

where the second identity is proved in [DdJO94, Corollary 4.5]. This contradicts the assumption that $\psi \in (\mathfrak{a}^i)^{\perp c}$. Therefore $\Phi_c(\phi)(\delta\psi) \neq 0$ and the first statement follows from Lemma 3.5. \square

Our ultimate goal is to show

Theorem 3.16. When $c > 1$, for all $i \geq 0$, $F_i^{\text{ind}} \mathbf{L}_c = F_i^\alpha \mathbf{L}_c$.

The containment in one direction is immediate:

Lemma 3.17. When $c > 1$, $F_i^{\text{ind}} \mathbf{L}_c \subset F_i^\alpha \mathbf{L}_c$

Proof. We do induction on c with respect to the same order as in the proof of Lemma 3.13. Assume $F_i^{\text{ind}} \mathbf{L}_{c'} \subset F_i^\alpha \mathbf{L}_{c'}$ holds for all $c' \prec c$ under the order (18). Take $\sum_k \xi_k \otimes \eta_k \in \mathbf{H}_c \delta e_- \otimes_{e_- \mathbf{H}_c e_-} e \mathbf{L}_{c-1}$ so that $\xi_k \in F_\alpha^{\text{ord}} \mathbf{H}_c \delta e_-$ and $\eta_k \in F_\beta^{\text{ind}} e \mathbf{L}_{c-1}$ and $\alpha + \beta \leq i$. We shall show $(\xi_k \otimes \eta_k) \perp_c \overline{\Phi}_c(\mathfrak{a}^{i+1})$.

By Lemma 3.5, it is sufficient to show that $p_{j_{i+1}} \cdots p_{j_1}(\xi_k \otimes \eta_k) = 0$ for $2 \leq j_1, \dots, j_{i+1} \leq n$. For this, we compute that

$$\begin{aligned} & p_{j_{i+1}} \cdots p_{j_1}(\xi_k \otimes \eta_k) \\ &= p_{j_{i+1}} \cdots p_{j_2}[p_{j_1}, \xi_k] \otimes \eta_k + p_{j_{i+1}} \cdots p_{j_2} \xi_k \otimes p_{j_1} \eta_k = \cdots \\ &= [p_{j_{i+1}}, [\cdots, [p_{j_{\alpha+1}}, [p_{j_\alpha}, [p_{j_{\alpha-1}}, [\cdots [p_{j_1}, \xi_k] \cdots]] \otimes \eta_k \\ &\quad + [p_{j_{i+1}}, [\cdots, [p_{j_{\alpha+1}}, [p_{j_\alpha}, [p_{j_{\alpha-1}}, [\cdots [p_{j_2}, \xi_k] \cdots]] \otimes p_{j_1} \eta_k + \cdots \\ &\quad + \xi_k \otimes p_{j_{i+1}} \cdots p_{j_1} \eta_k. \end{aligned}$$

Notice that $[p_{j_{i+1}}, [\cdots [p_{j_{\alpha+1}}, [p_{j_\alpha}, [p_{j_{\alpha-1}}, [\cdots [p_{j_{\ell+1}}, \xi_k] \cdots]] \in F_{\alpha-i+\ell-1}$ when $\alpha - i + \ell - 1 \geq 0$ and equals to 0 otherwise.

On the other hand, by the induction hypothesis, $\eta_k \perp \Phi_c(\mathfrak{a}^{\beta+1})\sigma_{c-1}$. As a result, $p_{j_\ell} \cdots p_{j_1} \eta_k \in F_{\beta-\ell}^{\mathfrak{a}} \mathfrak{e} L_{c-1}$ when $\ell \leq \beta$ and equals to 0 otherwise.

Since $\alpha + \beta \leq i$, we see that either $[p_{j_{i+1}}, [\cdots [p_{j_{\alpha+1}}, [p_{j_\alpha}, [p_{j_{\alpha-1}}, [\cdots [p_{j_{\ell+1}}, \xi_k] \cdots]] = 0$ or $p_{j_\ell} \cdots p_{j_1} \eta_k = 0$ and the lemma follows. \square

This gives a different proof of [GORS14, Theorem 4.8]. To show the opposite containment, our idea is to do induction on i . For the base case:

Lemma 3.18. *When $c > 1$, $F_0^{\text{ind}} L_c = F_0^{\mathfrak{a}} L_c$.*

Proof. It suffices to show $F_0^{\text{ind}} L_c \supset F_0^{\mathfrak{a}} L_c$. Under the isomorphism (16),

$$F_0^{\mathfrak{a}} L_c \stackrel{\text{Lemma 2.6}}{=} \overline{\Phi}_c(\mathfrak{a}^{\perp c}) = \overline{\Phi}_c(\mathcal{H}_c) = \Phi_c(\mathbb{C}[y_1 - y_2, \dots, y_{n-1} - y_n] \delta) \mathfrak{e}^\mu = \mathbb{C}[x_1, \dots, x_n] \Phi_c(\delta) \mathfrak{e}^\mu$$

is mapped to

$$\begin{aligned} & \text{Im}(\mathbb{C}[x_1, \dots, x_n] \mathfrak{e}_- \otimes \mathfrak{e}^{\mu - \frac{n(n-1)}{2}} \mathfrak{H}_c \mathfrak{e}_- \otimes_{\mathfrak{e} \mathfrak{H}_{c-1} \mathfrak{e}} \mathfrak{e} L_{c-1}) \\ & \subset \text{Im}(F_0^{\text{ord}}(\mathfrak{H}_c \mathfrak{e}_-) \otimes F_0^{\text{ind}} \mathfrak{e} L_{c-1} \rightarrow \mathfrak{H}_c \mathfrak{e}_- \otimes_{\mathfrak{e} \mathfrak{H}_{c-1} \mathfrak{e}} \mathfrak{e} L_{c-1}) = F_0^{\text{ind}} L_c. \end{aligned} \quad \square$$

This recovers [GORS14, Corollary 4.11] from a different point of view.

Example 3.19. (See Figure 1) Consider $L_{\frac{4}{3}}$ whose dimension is $4^3 - 1 = 16$. We compute $F^{\mathfrak{a}}, F^{\text{ind}}, F^{\text{alg}}$ on $L_{\frac{4}{3}}$. In this case, $\mathcal{H}_c = \mathcal{H}_c^{\text{triv}} \oplus (\mathcal{H}_c^{\text{st}})^{\oplus 2} \oplus \mathcal{H}_c^{\text{sign}}$. Inside, $(\mathcal{H}_c^{\text{st}})^{\oplus 2} = \langle \xi_1, \xi_2 \rangle \oplus \langle \alpha_1, \alpha_2 \rangle$ where $\xi_i = x_i - x_{i+1}$ and $\alpha_i = (y_i - y_{i+1}) \delta$. By Lemma 3.18, we have $F_0^{\text{ind}} L_c = F_0^{\mathfrak{a}} L_c = \langle \overline{\Phi}_c(1), \overline{\Phi}_c(\xi_i), \overline{\Phi}_c(\alpha_i), \overline{\Phi}_c(\delta) \rangle$. Since $\overline{\Phi}_c(1), \overline{\Phi}_c(\xi_i), \overline{\Phi}_c(\alpha_i), \overline{\Phi}_c(\delta)$ are all highest weight vectors under the \mathfrak{sl}_2 triple $\{\mathfrak{e}, \mathfrak{f}, \mathfrak{h}\}$, Lemma 3.13 tells us that $\mathfrak{f}^i(v) \in F_i^{\text{ind}} L_c \cap F_i^{\mathfrak{a}} L_c$ for any $v \in F_0^{\mathfrak{a}} L_c$.

It remains to determine where p_3 belongs to. Since $p_3 \in (\mathfrak{a}^2)^{\perp c}$ and is not c -harmonic, $\overline{\Phi}_c(p_3) \in F_1^{\mathfrak{a}} L_c$. On the other hand, by Lemma 3.1, $p_3 = \frac{1}{9} \sum_{i=1}^2 (x_i - x_{i+1})(y_i - y_{i+1}) p_3$. Because $(y_i - y_{i+1}) p_3 \in \mathcal{H}_c$, we conclude that $\overline{\Phi}_c(p_3) = \frac{1}{9} \sum_{i=1}^2 (y_i - y_{i+1}) \overline{\Phi}_c((y_i - y_{i+1}) p_3) \in F_1^{\text{ind}} L_c$.

3.3.1 Orthogonal lifts and the orthogonal complement of I_{c-1}^W

The product $\mathcal{H}_c \cdot \mathbb{C}[\mathfrak{h}]_{\leq i}^W$ denotes the vector space generated by all $h\psi$ with $h \in \mathcal{H}_c$ and $\psi \in \mathbb{C}[\mathfrak{h}]_{\leq i}^W$. Since $\mathcal{H}_c \cdot \mathbb{C}[\mathfrak{h}]_{\leq i}^W \cong \mathbb{C}[\mathfrak{h}] / \mathfrak{a}^{i+1}$, there is a unique $\underline{h\psi} \in L_c$ so that $\underline{h\psi} \in (\mathfrak{a}^{i+1})^{\perp c}$ and $\underline{h\psi} \equiv h\psi$ modulo \mathfrak{a}^{i+1} . We will use this notation frequently from now on. In the sequel, all polynomials will be homogeneous.

Write $(\mathfrak{a}^j)_{j-1}^{\perp c} := (\mathfrak{a}^j)^{\perp c} \cap \mathfrak{a}^{j-1}$. Then when $j \neq j'$

$$(\mathfrak{a}^{j+1})_j^{\perp c} \perp_c (\mathfrak{a}^{j'+1})_{j'}^{\perp c}.$$

Inside $(\mathfrak{a}^{j+1})_j^{\perp c}$, consider the subspace

$$\mathcal{S}_j := \langle \underline{h\psi} \in (\mathfrak{a}^{j+1})_j^{\perp c} \mid h \in \mathcal{H}_c, \psi \in I_{c-1}^W \rangle. \quad (20)$$

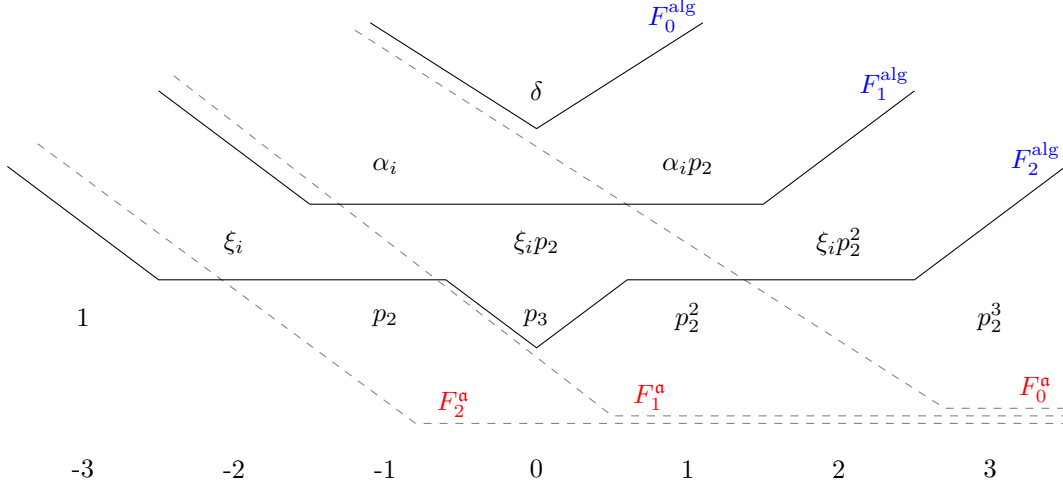


Figure 1: Filtrations from example 3.19 where the numbers at the bottom indicate \mathbf{h} -weights

We can go through the Gram-Schmidt process to obtain an orthogonal basis under the nondegenerate form $(-, -)_{c-1}$ on $((I_{c-1})^{\perp c} \cap (\mathfrak{a}^{j+1})_j^{\perp c})^W \pmod{I_c}$, which we denote by $\{\widehat{\psi}\}$. Also, we take an orthogonal basis $\{\widehat{h}\}$ of \mathcal{H}_c such that $\widehat{h} = \widehat{P}\delta$ with $\widehat{P} \in \mathbb{C}[y_1 - y_2, \dots, y_{n-1} - y_n]$ depending on \widehat{h} .

Define $\mathcal{S}_{j, \perp c} = \langle \underline{h\psi} \in (\mathfrak{a}^{j+1})_j^{\perp c} \mid h \in \mathcal{H}_c, \psi \in (I_{c-1}^{\perp c})^W \rangle$.

Lemma 3.20. *For any integer $j \geq 0$, suppose $F_j^{\text{ind}}\mathfrak{eL}_{c-1} = F_j^{\mathfrak{a}}\mathfrak{eL}_{c-1}$. Then $\overline{\Phi}_c(\mathcal{S}_{j, \perp c}) \subset F_j^{\text{ind}}\mathfrak{L}_c$.*

Proof. For orthogonal basis elements $\widehat{\psi}$ of $((I_{c-1})^{\perp c} \cap (\mathfrak{a}^{j+1})_j^{\perp c})^W$ and $\widehat{h} = \widehat{P}\delta$ of \mathcal{H}_c chosen as above, we claim the following

- (a) $\widehat{P}(\delta\widehat{\psi}) \in (\mathfrak{a}^{j+1})^{\perp c}$.
- (b) $\overline{\Phi}_c(\widehat{P}(\delta\widehat{\psi})) \in F_j^{\text{ind}}\mathfrak{L}_c$.
- (c) For any pair $(\widehat{h}', \widehat{\psi}')$, $(\widehat{P}(\delta\widehat{\psi}), \widehat{h}'\widehat{\psi}')_c \neq 0$ if and only if $\widehat{h}' = \widehat{h}$ and $\widehat{\psi}' = \widehat{\psi}$.

Here (a) holds because $\delta\widehat{\psi} \in (\mathfrak{a}^{j+1})^{\perp c}$ by Lemma 3.15 and so does $\widehat{P}(\delta\widehat{\psi})$.

For (b), by Lemma 3.3, $\overline{\Phi}_c(\delta\widehat{\psi}) = \Phi_c(\delta\widehat{\psi})\beta_c$ for some nonzero highest weight vector β_c in \mathfrak{L}_c , which is mapped to $\Phi_{c-1}(\widehat{\psi})\beta_{c-1}$ for some nonzero highest weight vector β_{c-1} in \mathfrak{L}_{c-1} under the isomorphism $\mathfrak{e}_-\mathfrak{L}_c \cong \delta\mathfrak{eL}_{c-1}$. The element $\Phi_{c-1}(\widehat{\psi})\beta_{c-1}$ lies in $F_j^{\mathfrak{a}}\mathfrak{eL}_{c-1}$ and hence belongs to $F_j^{\text{ind}}\mathfrak{L}_{c-1}$ by the assumption of the lemma. Therefore so does $\overline{\Phi}_c(\widehat{P}(\delta\widehat{\psi})) = \Phi_c(\widehat{P})\overline{\Phi}_c(\delta\widehat{\psi})$.

To show (c), notice that $\Phi_c(\delta\widehat{\psi}')(\delta\widehat{\psi}) = (\delta, \delta)_c(\widehat{\psi}', \widehat{\psi})_{c-1} \neq 0$ if and only if $\widehat{\psi} = \widehat{\psi}'$. Therefore $\Phi_c(\widehat{\psi}')(\delta\widehat{\psi}) \neq 0$ if and only if $\widehat{\psi} = \widehat{\psi}'$, in which case

$$(\widehat{P}(\delta\widehat{\psi}), \widehat{h}'\widehat{\psi}')_c = (\widehat{P}\Phi_c(\widehat{\psi})(\delta\widehat{\psi}), \widehat{h}')_c = (\widehat{\psi}, \widehat{\psi})_{c-1}(\widehat{P}(\delta), \widehat{h}')_c = (\widehat{\psi}, \widehat{\psi})_{c-1}(\widehat{h}, \widehat{h}')_c \neq 0 \text{ iff } \widehat{h} = \widehat{h}'.$$

As a consequence, $\{\overline{\Phi}_c(\widehat{P}(\delta\widehat{\psi}))\}$ forms a new basis of $\overline{\Phi}_c(\mathcal{S}_{j, \perp c})$ that is contained in $F_j^{\text{ind}}\mathfrak{L}_c$. Hence the lemma follows. \square

Lemma 3.21. $\mathcal{S}_{j, \perp c} = (\mathcal{S}_j)^{\perp c} \cap (\mathfrak{a}^{j+1})_j^{\perp c}$.

Proof. By Lemma 2.7, $\mathcal{S}_{j, \perp c}$ and \mathcal{S}_j together span $(\mathfrak{a}^{j+1})_j^{\perp c}$. Therefore we only need to show $\mathcal{S}_{j, \perp c} \subset (\mathcal{S}_j)^{\perp c} \cap (\mathfrak{a}^{j+1})_j^{\perp c}$. To do so, by the proof of Lemma 3.20, it suffices to show that $(\widehat{P}(\delta\widehat{\psi}), \underline{h\phi})_c = 0$ for all $\underline{h\phi} \in (\mathfrak{a}^{j+1})_j^{\perp c}$ such that $h \in \mathcal{H}_c$ and $\phi \in I_{c-1}^W$.

Because of Lemma 2.4, we have that $(\widehat{P}(\delta\widehat{\psi}), h\phi)_c = (\delta\widehat{\psi}, \Phi_c(\widehat{P})h\phi)_c$ is nonzero if and only if there exists some $\phi^\circ \in \mathbb{C}[\mathfrak{h}]^W$ so that

$$(\delta\widehat{\psi}, \delta\phi^\circ)_c = (\delta\widehat{\psi}, \Phi_c(\widehat{P})h\phi)_c.$$

By Lemma 2.4, $\delta\phi^\circ \in I_{c-1}^{\text{sign}}$. Moreover, note that $I_{c-1}^{\text{sign}} = \delta I_{c-1}^W$ and hence $\phi^\circ \in I_{c-1}$. We conclude the lemma by

$$(\delta\widehat{\psi}, \Phi_c(\widehat{P})h\phi)_c = (\delta, \delta)_c(\widehat{\psi}, \phi^\circ)_{c-1} = 0. \quad \square$$

3.3.2 Criteria for induction

Lemma 3.20 tells us that we can show $F^{\text{ind}} = F^\mathfrak{a}$ on $\mathcal{S}_{j,\perp c}$ by induction on c . As for \mathcal{S}_j , with doing induction on indexes of the filtrations in question, we prove Proposition 3.23.

Lemma 3.22. (i) *The quotient map $\cup_j \mathcal{S}_{j,\perp c} \rightarrow \mathbb{L}_c / (I_{c-1}^W)$ is an isomorphism.*

(ii) $\text{Im}(F_0^{\text{ord}}(\mathbb{H}_c e_-) \otimes_{\mathbb{C}} e\mathbb{L}_{c-1} \rightarrow \mathbb{H}_c e_- \otimes_{e\mathbb{H}_{c-1} e} e\mathbb{L}_{c-1} \cong \mathbb{L}_c) = \mathbb{L}_c / (I_{c-1}^W)$.

Proof. (i) follows directly from the definition of $\mathcal{S}_{j,\perp c}$. To see (ii), note that $(\mathbb{L}_c / (I_{c-1}^W))^W = e\mathbb{L}_c / I_{c-1}^W = e\mathbb{L}_{c-1}$ and so by Lemma 2.7,

$$\mathbb{L}_c / (I_{c-1}^W) = \mathcal{H}_c \cdot (\mathbb{L}_c / (I_{c-1}^W))^W.$$

Moreover, $F_0^{\text{ord}}(\mathbb{H}_c e_-) \cong \mathbb{C}[x_1, \dots, x_n]$. It remains to notice that

$$\mathbb{C}[x_1, \dots, x_n] \cdot e\mathbb{L}_{c-1} = (\mathbb{C}[x_1, \dots, x_n] / \mathfrak{a}) \cdot e\mathbb{L}_{c-1} = \mathcal{H}_c \cdot \mathbb{L}_c^W / I_{c-1}^W. \quad \square$$

Proposition 3.23. *For $j \geq 1$, suppose $F_{j-1}^{\text{ind}} \mathbb{L}_c = F_{j-1}^\mathfrak{a} \mathbb{L}_c$. Then the following statements are equivalent*

(i) *For any nonzero $\psi \in \mathcal{S}_j$ and $k \in [1, n-1]$ such that $(y_k - y_{k+1})\psi \notin (\mathfrak{a}^j)^{\perp c}$, $\overline{\Phi}_c(\psi) = 0$.*

(ii) *For any nonzero $\phi \in \mathcal{S}_j$ such that $(y_k - y_{k+1})\phi \in \mathfrak{a}^j$ for all $k \in [1, n-1]$, $\overline{\Phi}_c(\phi) = 0$.*

(iii) $\overline{\Phi}_c(\mathcal{S}_j \cap (\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})^{\perp c}) = 0$.

(iv) $F_j^{\text{ind}} \mathcal{S}_j = F_j^\mathfrak{a} \mathcal{S}_j$.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Leftrightarrow (iii): The property (7) $((x_i - x_j)\phi, \psi)_c = (\phi, (y_i - y_j)\psi)_c$ implies that $\phi \in (\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})^{\perp c}$ is equivalent to $(y_k - y_{k+1})\phi \perp_c (\mathfrak{a}^j)^{\perp c}$.

(iii) \Rightarrow (i): First of all, note that $\overline{\Phi}_c(\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})$ is a subspace of $\overline{\Phi}_c((\mathfrak{a}^{j+1})^{\perp c})$. Let

$$pr : \overline{\Phi}_c((\mathfrak{a}^{j+1})^{\perp c}) \rightarrow \overline{\Phi}_c(\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})$$

denote the orthogonal projection. Then because of (7), (i) is equivalent to the condition that $\psi \neq pr(\psi)$. Take $\phi = \psi - pr(\psi)$. Then ϕ lies in $((\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})^{\perp c}) \cap (\mathfrak{a}^{j+1})^{\perp c}$, which equals to 0 by assumption (ii).

(iv) \Rightarrow (iii): Take $\phi \in \mathcal{S}_j$ such that $\overline{\Phi}_c(\phi) \in F_j^\mathfrak{a} \mathbb{L}_c = F_j^{\text{ind}} \mathbb{L}_c$ with $j > 0$. By the definition of F^{ind} , $\overline{\Phi}_c(\phi) = \sum_{\alpha+\beta=j} \phi_\alpha \otimes \phi_\beta$ where $\phi_\alpha \in F_\alpha^{\text{ord}} \mathbb{H}_c e_-$ and $\phi_\beta \in F_\beta^{\text{ind}} e\mathbb{L}_{c-1}$. Since the image of ϕ in $\mathbb{L}_c / (I_{c-1}^W)$ is 0, given Lemma 3.22, we may assume $\alpha > 0$. As a result, $\overline{\Phi}_c(\phi)$ can be written in the form of $\sum_k (y_k - y_{k+1}) \overline{\Phi}_c(\phi_k)$ for some $\overline{\Phi}_c(\phi_k) \in F_{j-1}^{\text{ind}}$ which equals to $F_{j-1}^\mathfrak{a}$ by our assumption. Therefore $\overline{\Phi}_c(\phi) \in \overline{\Phi}_c(\sum_{k=1}^n x_k (\mathfrak{a}^j)^{\perp c})$.

(i) \Rightarrow (iv): When $\overline{\Phi}_c(\psi)$ is a lowest weight vector, which lies $F_\mu^\mathfrak{a}$, we have already shown $\psi \in F_\mu^{\text{ind}}$ by Lemma 3.13. Hence below we assume $\overline{\Phi}_c(\psi)$ is of weight $\nu > -\mu$.

Then from $\mathfrak{h}\underline{\psi} = \nu\underline{\psi}$ and Lemma 3.1 we deduce

$$\sum_{i < j} (x_i - x_j)(y_i - y_j)\underline{\psi} = n(\nu + \mu)\underline{\psi}.$$

Therefore

$$\overline{\Phi}_c(\underline{\psi}) = \frac{1}{n(\nu + \mu)} \sum (y_i - y_j) \overline{\Phi}_c((y_i - y_j)\underline{\psi}). \quad (21)$$

By assumption, $\overline{\Phi}_c((y_i - y_j)\underline{\psi}) \in F_{j-1}^\mathfrak{a} \mathbb{L}_c = F_{j-1}^{\text{ind}} \mathbb{L}_c$ and hence $\overline{\Phi}_c(\underline{\psi}) \in F_j^{\text{ind}} \mathbb{L}_c$. Thus (iv) \Rightarrow (i) is proved. \square

When $\psi \in \mathfrak{eL}_c$, we have $(y_k - y_{k+1})\psi \in (\mathfrak{a}^j)^{\perp c}$ by Lemma 3.6. As a consequence of Proposition 3.23, to show $F^{\text{ind}} = F^{\mathfrak{a}}$ we only need to consider elements in the form of $\sum \underline{h}\psi$ where $\psi \in \mathbb{C}[\mathfrak{h}]^W$ and $h \in \mathcal{H}_c$ is not a constant.

Proposition 3.23 also suggests that: to do induction, it is sufficient to show $(y_k - y_{k+1})\psi \in (\mathfrak{a}^j)^{\perp c}$ for all k if $\psi \in (\mathfrak{a}^{j+1})^{\perp c}$. This is our goal in the next section.

4 The kernel of the Dunkl form and the coinvariant algebra

4.1 The kernel of the Dunkl form

Throughout we assume $c = \frac{m}{n} > 1$ for positive integer m coprime to n . Define u_i to be the elementary symmetric polynomials satisfying $\prod_{i=1}^n (1 - x_i z) = \sum u_i z^i$ (put $u_1 = 0$). Let $v_i^{(c)}$ be the polynomials such that the formal Taylor expansion of the following equation holds:

$$\left(\sum u_i z^i \right)^c = \sum v_i^{(c)} z^i.$$

For $i = 1, \dots, n$, define

$$\begin{aligned} f_i &= \text{Coef}_z^m \left((1 - x_i z)^{-1} \prod_{k=1}^n (1 - x_k z)^c \right) = \sum_{j=0}^m x_i^j v_{m-j}^{(c)} \\ &= \text{Coef}_z^m \left(\prod_{l \neq i} (1 - x_l z) \prod_{k=1}^n (1 - x_k z)^{c-1} \right) = \sum_{j=0}^{n-1} \left((-1)^j \sum_{l_1 < \dots < l_j, l_1, \dots, l_j \neq i} x_{l_1} \cdots x_{l_j} \right) v_{m-j}^{(c-1)}. \end{aligned}$$

Lemma 4.1. *For $k < n$, one has*

$$x_i^k + x_i^{k-2} u_2 + \cdots + u_k = (-1)^k \sum_{j_1 < \dots < j_k, j_\ell \neq i} x_{j_1} \cdots x_{j_k}.$$

As a consequence, $x_i^k = (-1)^k \sum_{j_1 < \dots < j_k, j_\ell \neq i} x_{j_1} \cdots x_{j_k} \pmod{\mathfrak{a}}$.

Proof. We do induction on k . When $k = 1$, it follows from $x_1 + x_2 + \cdots + x_n \in \mathfrak{a}$. When $k > 1$, assume the lemma is proved for integers less than k . Then by the induction hypothesis

$$\begin{aligned} x_i^k &= x_i \left((-1)^{k-1} \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq i} x_{j_1} \cdots x_{j_{k-1}} - (x_i^{k-3} u_2 + \cdots + u_{k-1}) \right) \\ &= (-1)^{k-1} \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq i} x_i x_{j_1} \cdots x_{j_{k-1}} - (x_i^{k-2} u_2 + \cdots + x_i u_{k-1}) \\ &= (-1)^{k-1} \left(u_k - \sum_{j_1 < \dots < j_k, j_\ell \neq i} x_{j_1} \cdots x_{j_k} \right) - (x_i^{k-2} u_2 + \cdots + x_i u_{k-1}) \\ &= (-1)^k \sum_{j_1 < \dots < j_k, j_\ell \neq i} x_{j_1} \cdots x_{j_k} - (x_i^{k-2} u_2 + \cdots + u_k). \quad \square \end{aligned}$$

Therefore

$$f_i = \sum_{j=0}^{n-1} (x_i^j + x_i^{j-2} u_2 + \cdots + u_j) v_{m-j}^{(c-1)}. \quad (22)$$

According to [Gor13, (4.2)], the summand $v_m^{(c-1)}$ in f_i can be expressed by

$$v_m^{(c-1)} = (2(c-1) - (m-2))v_{m-2}^{(c-1)} u_2 + \cdots + ((n-1)(c-1) - (m-n-1))v_{m-n+1}^{(c-1)} u_{n-1}.$$

Suppose $kn < m < (k+1)n$. Then $v_{m-l}^{(c-1)} \in \mathfrak{a}^k$ for $l = m - nk, \dots, n-1$, and $v_{m-l}^{(c-1)} \in \mathfrak{a}^{k+1}$ when $l = 0, \dots, m - nk - 1$. Hence

$$f_i \equiv \sum_{j=m-nk}^{n-1} x_i^j v_{m-j}^{(c-1)} \pmod{\mathfrak{a}^{k+1}}. \quad (23)$$

Theorem 4.2. [Dun98, Chapter 5][CE03, Proposition 3.1][Gor13, Theorem 4.3] We have that

- (i) $I_c = \sum_i \mathbb{C}[\mathfrak{h}]f_i$ and $\sum_i \mathbb{C}f_i$ form the standard permutation representation of S_n .
- (ii) $\sum x_i^\ell f_i = -\frac{m+\ell}{c} v_{m+\ell}^{(c)} - \sum_{j=1}^{\ell-1} v_{m+j}^{(c)} p_{\ell-j}$.
- (iii) $I_c^W = (v_{m+1}^{(c)}, \dots, v_{m+n-1}^{(c)})$ and $v_k^{(c)} \in I_c^W$ for $k > m+n-1$.

Given the expression (22) and (iii) of Theorem 4.2, we see that I_c is contained in the ideal generated by I_{c-1}^W .

Remark 4.3. The generators $v_{m-n+1}^{(c-1)}, \dots, v_{m-1}^{(c-1)}$ of I_{c-1}^W form a regular sequence. As a result, $\dim(\mathbb{L}_c/(I_{c-1}^W)) = (m-n+1) \cdots (m-1) = \dim \mathcal{H}_c \cdot \dim(\mathbb{e}_{\mathbb{L}_{c-1}})$, which implies that $\mathbb{L}_c/(I_{c-1}^W) \cong \mathcal{H}_c \otimes_{\mathbb{C}} \mathbb{e}_{\mathbb{L}_{c-1}}$.

4.2 Equality on F_1

In this section and the next section, we will show $F_1^{\text{ind}} = F_1^{\mathfrak{a}}$. In the process we will acquire all the ingredients needed for proving $F_j^{\text{ind}} = F_j^{\mathfrak{a}}$ for $j \geq 1$.

Take a nonzero polynomial $\phi \in (\mathfrak{a}^2)^{\perp c} \cap \mathfrak{a}$. Assume $\phi \equiv \sum_k h_k p_k \pmod{\mathfrak{a}^2}$ for $h_k \in \mathcal{H}_c$. By Lemma 3.20 and 3.21, we may assume, in addition, $p_k \in \overline{I_{c-1}^W} \pmod{\mathfrak{a}^2}$. From Theorem 4.2, we know that $I_{c-1}^W \pmod{\mathfrak{a}^2}$ is generated by $v_{m-n+1}^{(c)}, \dots, v_n^{(c)}$. Therefore to have $I_{c-1}^W \neq 0 \pmod{\mathfrak{a}^2}$, the integer m has to satisfy $n < m < 2n$. We will assume $n < m < 2n$ for the rest of Section 4.2 and Section 4.3.

Now to show $F_1^{\text{ind}} = F_1^{\text{alg}}$, by Lemma 3.18, Lemma 3.20 and Proposition 3.23, we only need to show that if (a lift of) $\phi := \sum_{k=m-n+1}^n h_k p_k$ lying in $\mathbb{C}[\mathfrak{h}] \cap (I_{c-1}^W)$ is such that $(y_i - y_{i+1})\phi \in \mathfrak{a}$ for all i , then $\phi \in I_c$. Here $h_{m-n+1}, \dots, h_n \in \mathcal{H}_c$.

We know that any element in I_c has the form

$$\sum_{i=1}^n \phi_i f_i = \sum_{i=1}^n \phi_i \sum_{j=0}^m x_i^{m-j} v_j^{(c)}. \quad (24)$$

Replacing f_i for $i = 1, \dots, n$ by

$$f_i \equiv x_i^{n-1} v_{m-n+1}^{(c-1)} + \cdots + x_i^{m-n} v_n^{(c-1)} \pmod{\mathfrak{a}^2}$$

and v_j for $j = m-n+1, \dots, n$ by

$$v_j^{(c)} \equiv c u_j \pmod{\mathfrak{a}^2}$$

we have the further expansion

$$\sum_{i=1}^n \phi_i f_i \equiv (c-1) \sum_{i=1}^n \phi_i \sum_{j=m-n+1}^n x_i^{m-j} u_j \pmod{\mathfrak{a}^2}. \quad (25)$$

Let $\mathcal{R} := \mathbb{C}[\mathfrak{h}]/\mathfrak{a}$ be the coinvariant algebra. Using row vector multiplication, we have two linear maps

$$\mathcal{R}^n \xrightarrow{A_{m-n,n-1}} \mathcal{R}^{2n-m} \xrightarrow{B_{m-n,n-1}} \mathcal{R}^n \quad (26)$$

defined by the following matrices

$$A_{m-n,n-1} := \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^{m-n} \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^{m-n} \\ & & \cdots & \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^{m-n} \end{pmatrix}, \quad B_{m-n,n-1} := \begin{pmatrix} x_1^{m-n} & x_2^{m-n} & \cdots & x_n^{m-n} \\ x_1^{m-n+1} & x_2^{m-n+1} & \cdots & x_n^{m-n+1} \\ & & \cdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}. \quad (27)$$

We will simply write $A := A_{m-n,n-1}$, $B := B_{m-n,n-1}$ when there is no ambiguity.

Lemma 4.4. (a) $\sum_{m-n+1}^n h_k p_k \in I_c$ holds if and only if $((m-n+1)h_{m-n+1}, \dots, nh_n) \in \text{Im}(A)$.

(b) $(y_i - y_{i+1}) \sum_{k=m-n+1}^n h_k p_k \in \mathfrak{a}$ holds for all i if and only if $((m-n+1)h_{m-n+1}, \dots, nh_n) \in \text{Ker}(B)$.

Proof. by Newton's identity $p_k \equiv k u_k \pmod{\mathfrak{a}^2}$, for $k = 2, \dots, n$. The equation

$$\sum h_k p_k \equiv \sum k h_k u_k = \sum \phi_i f_i \equiv (c-1) \sum_{i=1}^n \phi_i \sum_k x_i^{m-k} k u_k = (c-1) \sum_k \left(\sum_{i=1}^n \phi_i x_i^{m-k} \right) u_k \pmod{\mathfrak{a}^2}.$$

then implies $((m-n+1)h_{m-n+1}, \dots, nh_n) = (c-1)(\phi_1 \dots \phi_n)A$. This proves (a).

Because p_k is symmetric, by [Gor13, Corollary 5.3] we have that

$$(y_i - y_{i+1})(h_k p_k) = ((y_i - y_{i+1})h_k)p_k + h_k((y_i - y_{i+1})p_k).$$

The element $(y_i - y_{i+1}) \sum_k h_k y_k$ is in \mathfrak{a} if and only if $\sum_k h_k ((y_i - y_{i+1})p_k)$ is in \mathfrak{a} . The latter element is equal to $\sum_k k h_k (x_i^{k-1} - x_{i+1}^{k-1})$, because $y_i - y_{i-1}$ acts on symmetric polynomials by $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}$.

This can be translated to

$$((m-n+1)h_{m-n+1}, \dots, nh_n) \begin{pmatrix} x_1^{m-n} - x_2^{m-n} & x_2^{m-n} - x_3^{m-n} & \dots & x_{n-1}^{m-n} - x_n^{m-n} \\ x_1^{m-n+1} - x_2^{m-n+1} & x_2^{m-n+1} - x_3^{m-n+1} & \dots & x_{n-1}^{m-n+1} - x_n^{m-n+1} \\ & & \dots & \\ x_1^{n-1} - x_2^{n-1} & x_2^{n-1} - x_3^{n-1} & \dots & x_{n-1}^{n-1} - x_n^{n-1} \end{pmatrix} = 0 \pmod{\mathfrak{a}}.$$

We add in a new column to the matrix in the last equation to define

$$B' = \begin{pmatrix} x_1^{m-n} - x_2^{m-n} & \dots & x_{n-1}^{m-n} - x_n^{m-n} & p_{m-n} \\ x_1^{m-n+1} - x_2^{m-n+1} & \dots & x_{n-1}^{m-n+1} - x_n^{m-n+1} & p_{m-n+1} \\ & & \dots & \\ x_1^{n-1} - x_2^{n-1} & \dots & x_{n-1}^{n-1} - x_n^{n-1} & p_{n-1} \end{pmatrix}.$$

As the last column in B' is 0 mod \mathfrak{a} , $((m-n+1)h_{m-n+1}, \dots, nh_n)B = 0 \pmod{\mathfrak{a}}$ is equivalent to $((m-n+1)h_{m-n+1}, \dots, nh_n)B' = 0 \pmod{\mathfrak{a}}$. Finally notice that B can be obtained from B' by elementary column operations. Therefore, $\text{Ker}(B) = \text{Ker}(B')$. \square

4.3 The coinvariant algebra

In this subsection, we prove that the sequence (26) is exact at the middle term.

Composing A and B gives

$$AB = \begin{pmatrix} nx_1^{m-1} & \sum_{j=m-n}^{n-1} x_1^j x_2^{m-j-1} & \dots & \sum_{j=m-n}^{n-1} x_1^j x_n^{m-j-1} \\ \sum_{j=m-n}^{n-1} x_2^j x_1^{m-j-1} & nx_2^m & \dots & \sum_{j=m-n}^{n-1} x_2^j x_n^{m-j-1} \\ & & \dots & \\ \sum_{j=m-n}^{n-1} x_1^j x_n^{m-j-1} & & \dots & nx_n^m \end{pmatrix}.$$

We will show $AB = 0 \pmod{\mathfrak{a}}$ below, which implies $\text{Im}(A) \subset \text{Ker}(B)$ for $n < m < 2n$.

Lemma 4.5. For $i \neq j$ and $k < n-1$, we have

$$x_i^k + x_i^{k-1} x_j + \dots + x_i x_j^{k-1} + x_j^k = (-1)^k \sum_{j_1 < \dots < j_k, j_\ell \neq i, j} x_{j_1} \dots x_{j_k} \pmod{\mathfrak{a}}.$$

Proof. We do induction on k . When $k = 1$, this follows from $x_1 + x_2 + \dots + x_n \in \mathfrak{a}$. Assume the lemma is proved for integer less than k where $k \geq 2$. By Lemma 4.1, we have

$$x_j^k = x_j((-1)^{k-1} \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq j} x_{j_1} \dots x_{j_{k-1}}) \pmod{\mathfrak{a}}. \quad (28)$$

On the other hand, notice that

$$- \sum_{j_1 < \dots < j_k, j_\ell \neq i, j} x_{j_1} \cdots x_{j_k} = x_i \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq i} x_{j_1} \cdots x_{j_{k-1}} + x_j \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq j} x_{j_1} \cdots x_{j_{k-1}} \pmod{\mathfrak{a}}. \quad (29)$$

Now by the induction hypothesis (modulo \mathfrak{a} below)

$$\begin{aligned} x_i^k &= x_i \left((-1)^{k-1} \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq i, j} x_{j_1} \cdots x_{j_{k-1}} - (x_i^{k-2} x_j + \dots + x_j^{k-1}) \right) \\ &= (-1)^{k-1} \sum_{j_1 < \dots < j_{k-1}, j_\ell \neq i, j} x_i x_{j_1} \cdots x_{j_{k-1}} - (x_i^{k-1} x_j + \dots + x_i x_j^{k-1}) \\ (\text{by (29)}) &= (-1)^k \left(\sum_{j_1 < \dots < j_k, j_\ell \neq i, j} x_{j_1} \cdots x_{j_k} + \left(\sum_{j_1 < \dots < j_{k-1}, j_\ell \neq j} x_j x_{j_1} \cdots x_{j_{k-1}} \right) \right) - (x_i^{k-1} x_j + \dots + x_i x_j^{k-1}) \\ (\text{by (28)}) &= (-1)^k \sum_{j_1 < \dots < j_k, j_\ell \neq i, j} x_{j_1} \cdots x_{j_k} - (x_i^{k-1} x_j + \dots + x_i x_j^{k-1} + x_j^k). \quad \square \end{aligned}$$

Corollary 4.6. For $m \geq n$, the element $\phi_{ij} := x_i^{m-n} x_j^{n-1} + x_i^{m-n+1} x_j^{n-2} + \dots + x_i^{n-1} x_j^{m-n}$ is in \mathfrak{a} . Thus, $AB = 0$, $\text{Im}(A) \subset \text{Ker}(B)$ and $\text{rank}(A) + \text{rank}(B) \leq (2n - m)n!$.

It remains to show $\text{rank}(A) + \text{rank}(B) = (2n - m)n!$. We first show the following lemma.

Lemma 4.7. $\text{rank}(A) = \text{rank}(B)$ and so $\text{rank}(A) \leq \frac{2n-m}{2}n!$.

Proof. Computing the rank of A by taking the span of its column vectors we obtain

$$\text{rank}(A) = \dim_{\mathbb{C}} \{ \phi_1(x_1^{n-1}, \dots, x_n^{n-1}) + \dots + \phi_{2n-m}(x_1^{m-n}, \dots, x_n^{m-n}), \phi_i \in \mathcal{R} \}.$$

Computing the rank of B by taking the span of its row vectors we obtain

$$\text{rank}(B) = \dim_{\mathbb{C}} \{ \psi_1(x_1^{m-n}, \dots, x_n^{m-n}) + \dots + \psi_{2n-m}(x_1^{n-1}, \dots, x_n^{n-1}), \psi_i \in \mathcal{R} \}.$$

As a result, $\text{rank}(A) = \text{rank}(B)$. □

It remains to show $\text{rank}(A) \geq \frac{2n-m}{2}n!$. Write $V_i := \mathcal{R}(x_i^{n-1}, \dots, x_i^{m-n})$, the \mathcal{R} -submodule in \mathcal{R}^{2n-m} generated by the i -th row vector of A . Then $\text{rank}(A) = \text{rank}(B) = \dim_{\mathbb{C}}(V_1 + \dots + V_n)$.

From now on, let \mathcal{R}_i denote the coinvariant algebra of i variables and $\text{Im}_n(x_i^{n-1}) := \text{Im}(x_i^{n-1}, \mathcal{R}_n \rightarrow \mathcal{R}_n)$. Note that the composition

$$\text{Im}_n(x_i^{n-1}) \xrightarrow{(x_i^{n-1})^T} \mathcal{R}_n \rightarrow \mathcal{R}_n / \text{Im}_n(x_i)$$

is an isomorphism and $\mathcal{R}_n / \text{Im}_n(x_i) \cong \mathcal{R}_{n-1}$. Let $\iota_i : \text{Im}_n(x_i^{n-1}) \rightarrow \mathcal{R}_{n-1}$ denote this isomorphism.

Then $\iota_1(\text{Im}_n(x_1^{n-1}) \cap \text{Im}_n(x_2^{n-1})) = \text{Ker}_{n-1}(x_2) = \text{Im}_{n-1}(x_2^{n-2}) \cong \mathcal{R}_{n-2}$.

When $m = 2n - 1$, $V_i = \text{Ker}(x_i) = \text{Im}(x_i^{n-1})$ and from the discussion above we deduce that $\dim(V_i) = (n - 1)!$ and $\dim(V_i \cap V_j) = (n - 2)!$ when $1 \leq i, j \leq n$ and $i \neq j$. In general:

Lemma 4.8. • When $n < m < 2n$, $\dim(V_i) = (2n - m)(n - 1)!$

• When $1 \leq i, j \leq n$ and $i \neq j$, $\dim(V_i \cap V_j) = (2n - m)(n - 2)!$

Proof. Define an operator $J_i : \mathcal{R}_n^{2n-m} \rightarrow \mathcal{R}_n^{2n-m}$ given by the following matrix using row multiplication

$$J_i := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -x_i & 1 & \cdots & 0 & 0 \\ 0 & -x_i & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -x_i & x_i^{2n-m} \end{pmatrix}$$

For example $J_i = x_i$ when $m = 2n - 1$. Then $V_i = \text{Ker}(J_i) \cong \text{Ker}(x_i^{2n-m})$ is of dimension $(2n - m)(n - 1)!$. Let $(J_i \ J_j)$ denote concatenation of the matrices J_i and J_j . When $i \neq j$, $\dim(V_i \cap V_j) = \dim \text{Ker}(J_i \ J_j)$. Using elementary matrix transformations, $(J_i \ J_j)$ can be transformed into

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & \cdots & x_i^{2n-m} & 0 & \cdots & x_i - x_j & x_j^{2n-m} \end{pmatrix}$$

Therefore, $\dim \text{Ker}(J_i \ J_j) = \dim \text{Ker}(x_i^{2n-m} \ x_i - x_j \ x_j^{2n-m})$.

Since $(x_i - x_j) | (x_i^{2n-m} - x_j^{2n-m})$, we get

$$\text{Ker}(x_i^{2n-m} \ x_i - x_j \ x_j^{2n-m}) = \text{Ker}(x_i^{2n-m} \ x_i - x_j).$$

Without loss of generality, we may assume $i = 1$ and $j = 2$.

By Lemma 4.9 below, one has $\text{Ker}(x_1 - x_2) = \sum_{j=1}^{n-1} \psi_j \mathcal{R}_{n-2}$ where

$$\psi_j := x_1^j x_2^{n-2} + 2x_1^{j+1} x_2^{n-3} + \cdots + (n-j)x_1^{n-1} x_2^{j-1} \in \text{Ker}(x_1 - x_2).$$

In particular, $\psi_{m-n}, \psi_{m-n+1}, \dots, \psi_{n-1} \in \text{Ker}(x_1^{2n-m})$. Since multiplication by x_1^{2n-m} preserves the basis

$x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}$ ($0 \leq a_i \leq n - i$) (possibly sending some elements to 0), we conclude that

$$\psi_{m-n} \mathcal{R}_{n-2} + \cdots + \psi_{n-1} \mathcal{R}_{n-2} = \text{Ker}(x_1^{2n-m}) \cap \text{Ker}(x_1 - x_2),$$

and so $\dim(V_1 \cap V_2) = \dim \text{Ker}(J_1 \ J_2) = \dim(\text{Ker}(x_1^{2n-m}) \cap \text{Ker}(x_1 - x_2)) = (2n - m)(n - 2)!$. \square

Lemma 4.9. *Let $\mathcal{R}_{n-2} \subset \mathcal{R}_n$ be generated by $x_3^{a_3} x_4^{a_4} \cdots x_{n-1}^{a_{n-1}}$, $0 \leq a_i \leq n - i$. For $1 \leq j \leq n - 1$, write*

$$\psi_j := x_1^j x_2^{n-2} + 2x_1^{j+1} x_2^{n-3} + \cdots + (n-j)x_1^{n-1} x_2^{j-1}$$

Then $\sum_{j=1}^{n-1} \psi_j \mathcal{R}_{n-2} = \text{Ker}(x_1 - x_2, \mathcal{R}_n \rightarrow \mathcal{R}_n)$.

Proof. We compute

$$\begin{aligned} & (x_1 - x_2)(x_1^j x_2^{n-2} + 2x_1^{j+1} x_2^{n-3} + \cdots + (n-j)x_1^{n-1} x_2^{j-1}) \\ &= -x_1^j x_2^{n-1} + x_1^{j+1} x_2^{n-2} - 2x_1^{j+1} x_2^{n-2} + 2x_1^{j+2} x_2^{n-3} + \cdots - (n-j)x_1^{n-1} x_2^j + (n-j)x_1^n x_2^{j-1} \\ &= -x_1^j x_2^{n-1} - x_1^{j+1} x_2^{n-2} - \cdots - x_1^{n-1} x_2^j. \end{aligned}$$

The element in the last line belongs to \mathfrak{a} by Corollary 4.6.

Next, consider the basis $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ ordered by lexicographic order on the powers. With respect to this basis, multiplication by x_1 can be expressed by

$$N \otimes I_{(n-1)!}$$

where I_k stands for the $k \times k$ identity matrix and

$$N := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}$$

Multiplication by x_2 can be expressed by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -N^{n-1} \\ I & 0 & \cdots & 0 & -N^{n-2} \\ 0 & I & \cdots & 0 & -N^{n-3} \\ & & \cdots & & \\ 0 & 0 & \cdots & I & -N \end{pmatrix} \otimes I_{(n-2)!}$$

Observe that, using elementary row operations, the following matrix that represents multiplication by $x_2 - x_1$

$$\begin{pmatrix} -N & 0 & \cdots & 0 & -N^{n-1} \\ I & -N & \cdots & 0 & -N^{n-2} \\ 0 & I & \cdots & 0 & -N^{n-3} \\ & & \cdots & & \\ 0 & 0 & \cdots & I & -2N \end{pmatrix} \otimes I_{(n-2)!}$$

can be transformed to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -nN^{n-1} \\ I & 0 & \cdots & 0 & -(n-1)N^{n-2} \\ 0 & I & \cdots & 0 & -(n-2)N^{n-3} \\ & & \cdots & & \\ 0 & 0 & \cdots & I & -2N \end{pmatrix} \otimes I_{(n-2)!}$$

By column operations we transform it to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -nN^{n-1} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \otimes I_{(n-2)!}$$

Since $\text{rank}(N^{n-1}) = 1$, we see $\dim(\text{Ker}(x_1 - x_2)) = (n-1)(n-2)!$. The lemma follows. \square

Lemma 4.10. $\dim(V_1 \cap (V_2 + \cdots + V_i)) \leq (i-1)(2n-m)(n-2)!$.

Proof. Define J_i as in the proof of Lemma 4.8 and for $\beta \geq 2$ put $J_{[2,\beta]} = J_2 J_3 \cdots J_\beta$. Consider the following short exact sequences:

$$\begin{aligned} 0 \rightarrow V_1 \cap V_2 \rightarrow V_1 \cap (V_2 + \cdots + V_i) \xrightarrow{J_2} (V_1 \cap (V_2 + \cdots + V_i))J_2 \rightarrow 0, \\ 0 \rightarrow [(V_1 \cap (V_2 + \cdots + V_i))J_2] \cap V_3 \rightarrow (V_1 \cap (V_2 + \cdots + V_i))J_2 \xrightarrow{J_3} (V_1 \cap (V_2 + \cdots + V_i))J_{[2,3]} \rightarrow 0, \\ \dots \dots \dots \\ 0 \rightarrow [(V_1 \cap (V_2 + \cdots + V_i))J_{[2,i-2]}] \cap V_i \rightarrow (V_1 \cap (V_2 + \cdots + V_i))J_{[2,i-2]} \xrightarrow{J_{i-1}} (V_1 \cap (V_2 + \cdots + V_i))J_{[2,i-1]} \rightarrow 0. \end{aligned}$$

We claim that when $j = 1, 2, \dots, i-1$

$$[(V_1 \cap (V_2 + \cdots + V_i))J_{[2,i-1]}] \cap V_j \subset [V_1 \cap \text{Ker}(J_{[2,j]})]J_{[2,j-1]} \subset V_1 \cap V_j. \quad (30)$$

First of all, $\text{Ker}(J_{[2,j]})J_{[2,j-1]} \subset \text{Ker}(J_j) = V_j$, from which the second containment follows. As for the first containment, take $v_1 = v_2 + \cdots + v_i \in V_1 \cap (V_2 + \cdots + V_i)$ where $v_j \in V_j$ for all j . If $v_1 J_{[2,j-1]} \in V_j$, then

$$(v_{j+1} + \cdots + v_i)J_{[2,j]} = 0,$$

i.e. $v_{j+1} + \cdots + v_i \in \text{Ker}(J_{[2,j]})$. As a result, $v_1 \in V_1 \cap (V_2 + \cdots + V_j + \text{Ker}(J_{[2,j]})) = V_1 \cap \text{Ker}(J_{[2,j]})$ and $v_1 J_{[2,j-1]} \in [V_1 \cap \text{Ker}(J_{[2,j]})]J_{[2,j-1]}$. Hence the claim is proved.

As a consequence of the short exact sequences and the claim,

$$\begin{aligned} & \dim(V_1 \cap (V_2 + \cdots + V_i)) \\ &= \dim(V_1 \cap V_2) + \dim((V_1 \cap (V_2 + V_3))J_2) + \cdots + \dim((V_1 \cap \text{Ker}(J_{[2,i]}))J_{[2,i-1]}) \\ &\leq \dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \cdots + \dim(V_1 \cap V_i). \end{aligned} \quad (31)$$

The last line equals to $(i-1)(2n-m)(n-2)!$ by Lemma 4.8. \square

Proposition 4.11. *We have a vector space decomposition:*

$$V_1 \cap (V_2 + \cdots + V_i) \cong (V_1 \cap V_2) \oplus \cdots \oplus (V_1 \cap V_i). \quad (32)$$

Proof. We compute

$$\begin{aligned}
\text{rank}(A) &= \dim(V_1 + \cdots + V_n) \\
&= \dim(V_1) + (\dim(V_2) - \dim(V_1 \cap V_2)) + \cdots + (\dim(V_n) - \dim(V_n \cap (V_1 + \cdots + V_{n-1}))) \\
&= n(2n-m)(n-1)! - (\dim(V_1 \cap V_2) + \dim(V_3 \cap (V_1 + V_2)) + \cdots + \dim(V_n \cap (V_1 + \cdots + V_{n-1}))) \\
&\geq n(2n-m)(n-1)! - \left((2n-m)(n-2)! + 2 \cdot (2n-m)(n-2)! + \cdots + (n-1) \cdot (2n-m)(n-2)! \right) \\
&= (2n-m) \frac{n(2n-2-(n-1))}{2} (n-2)! = \frac{2n-m}{2} n!
\end{aligned}$$

Here the second equality follows from Lemma 4.8 and the third equality follows from Lemma 4.10. By Corollary 4.7, we conclude that $\dim(V_1 + \cdots + V_n) = \frac{2n-m}{2} n!$. Therefore, (31) is actually an equality for $i = 1, \dots, n$, the inclusions (30) are equalities. Thus the short exact sequences in the proof of Lemma 4.10 yields (32). \square

Corollary 4.12. *We have $\dim_{\mathbb{C}}(V_1 + \cdots + V_k) = \frac{2n-m}{2} k(2n-1-k)n!$ for $1 \leq k \leq n$. In particular $\text{rank}(A) = \frac{2n-m}{2} n!$ and $\text{Im}(A) = \text{Ker}(B)$.*

Proof. We compute

$$\begin{aligned}
\dim(V_1 + \cdots + V_k) &= \dim(V_1) + (\dim(V_2) - \dim(V_1 \cap V_2)) + \cdots + (\dim(V_k) - \dim(V_k \cap (V_1 + \cdots + V_{k-1}))) \\
&= k(2n-m)(n-1)! - (\dim(V_1 \cap V_2) + \dim(V_3 \cap (V_1 + V_2)) + \cdots + \dim(V_k \cap (V_1 + \cdots + V_{k-1}))) \\
&= k(2n-m)(n-1)! - \left((2n-m)(n-2)! + 2 \cdot (2n-m)(n-2)! + \cdots + (k-1) \cdot (2n-m)(n-2)! \right) \\
&= (2n-m) \frac{k(2n-2-(k-1))}{2} (n-2)! = \frac{2n-m}{2} k(2n-1-k)n!
\end{aligned}$$

Remark 4.13. *The dimension count of Corollary 4.12 implies that $\text{Im}(A) = \text{Ker}(B)$ is a Lagrangian subspace in $H^*(\mathcal{B})^{\oplus(2n-m)}$ (where \mathcal{B} is the flag variety) with respect to the Poincare form defined by*

$$(\alpha, \beta) = d, \quad \text{if } \alpha \cup \beta = d \cdot \delta^{\oplus(2n-m)}.$$

Corollary 4.14. *If $\sum h_k p_k \in \mathcal{H}_c \cdot (I_{c-1}^W \cap \mathbb{C}[\mathfrak{h}]_1^W)$ and $(y_i - y_{i+1})(\sum h_k p_k) \in \mathfrak{a}$ holds for all i , then $\sum h_k p_k \in I_c$.*

Proof. By Lemma 4.4 and Corollary 4.12. \square

Corollary 4.15. *We have $F_1^{\text{ind}} L_c = F_1^{\mathfrak{a}} L_c$ for all $c > 1$.*

Proof. Lemma 3.17 says $F_1^{\text{ind}} L_c \subset F_1^{\mathfrak{a}} L_c$. Lemma 3.18 guarantees that the assumptions of Lemma 3.20 and Proposition 3.23 hold. Lemma 3.20 implies $\overline{\Phi}_c(\mathcal{S}_{1, \perp c}) \subset F_1^{\text{ind}} L_c$. Corollary 4.14 implies part (iii) of Proposition 3.23 and therefore $\overline{\Phi}_c(\mathcal{S}_1) \subset F_1^{\text{ind}} L_c$. \square

Example 4.16. (See figure 2) Consider $c = \frac{5}{3}$, where $\dim(L) = 5^{3-1} = 25$. Based on Example 3.19, it remains to determine where $p_3 \xi_i$ lies (recall that $\xi_i = x_i - x_{i+1}$). Note that $\Phi_c(p_2)(p_3 \xi_i) \in \mathfrak{e}_{\text{st}} L_c$ is of weight -2 and hence can not be symmetric. Therefore $\Phi_c(p_2^2)(p_3 \xi_i)$ must be 0 and thus $p_3 \xi_i \in (\mathfrak{a}^2)^{\perp c}$. Since $p_3 \in I_{\frac{5}{3}}^W$, it follows from Corollary 4.12 that $(y_k - y_{k+1})(p_3 \xi_i) \in \mathcal{H}_c$ for $k = 1, 2$. This is not obvious by direct computation.

4.3.1 Springer fiber at the minimal nilpotent orbit

There is a mysterious relationship between the matrices A, B and the minimal Springer fiber.

Let $G = \text{SL}_n$ with Lie algebra \mathfrak{g} and $B \subset G$ the Borel consisting of upper triangular matrices. Let \mathfrak{b} be the Lie algebra of B with nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

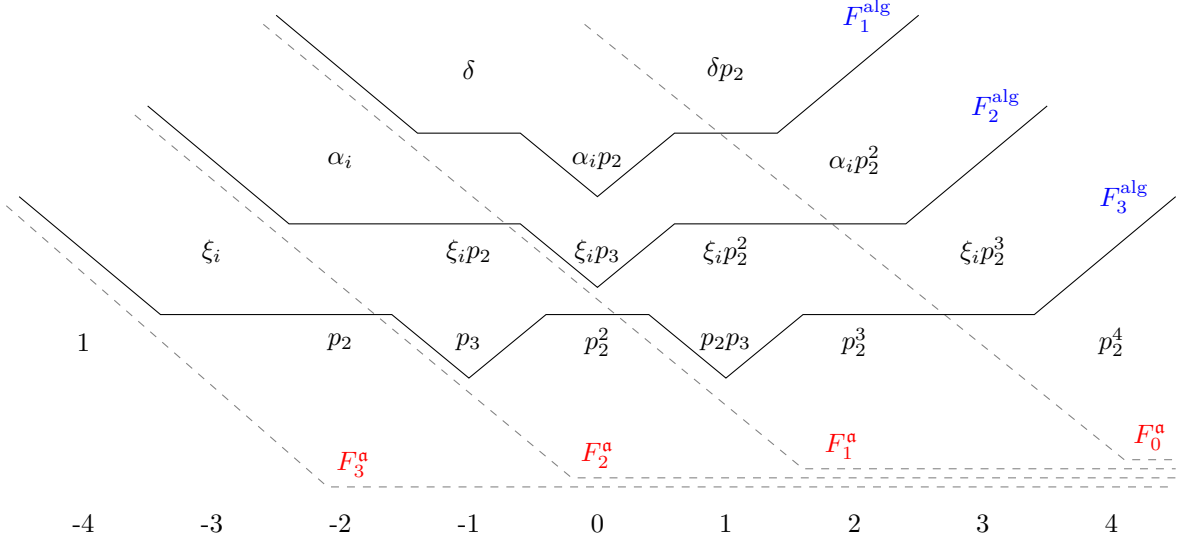


Figure 2: Filtrations from example 4.16 where the numbers on the bottom indicate \mathfrak{h} -weights

Let \mathcal{N} be the nilpotent cone in \mathfrak{g} , there is a unique minimal nilpotent G -orbit consisting of nilpotent elements in \mathfrak{g} of rank 1. Let e_{min} be an element of this orbit and \mathcal{B}_{min} the fiber of the Springer map $G \times^B \mathfrak{n} \rightarrow \mathcal{N}$ over e_{min} . By [DCP81] one knows that

$$H^*(\mathcal{B}_{min}) = \mathbb{C}[\mathfrak{h}] / [(\mathbb{C}[\mathfrak{h}]_+^W) + (x_1^{n-1}, \dots, x_n^{n-1})].$$

In the case of $m = 2n - 1$, the right hand side is exactly $\mathcal{R}/\text{Im}(A) = \mathcal{R}/\text{Ker}(B) \cong \text{Im}(B)$. Now because $H^*(\mathcal{B}_{min}) \cong \text{ind}_{S_2}^{S_n} \text{triv}$, its dimension is exactly $\frac{n!}{2}$. This gives a different proof of Corollary 4.12 when $m = 2n - 1$. We do not know how to generalize it to arbitrary $n < m < 2n$.

4.4 F_ℓ for $\ell \geq 1$

Recall the subspace $\mathcal{S}_\ell := \langle h\psi \in (\mathfrak{a}^{\ell+1})_\ell^{\perp c} \mid h \in \mathcal{H}_c, \psi \in I_{c-1}^W \cap \mathfrak{a}^\ell \rangle$ which satisfies $\overline{\Phi}_c(\mathcal{S}_\ell) \subset F_\ell^a L_c$. Our goal in this section is to show that $\overline{\Phi}_c(\mathcal{S}_\ell) \subset F_\ell^{\text{ind}} L_c$. The following lemma is the easiest case of Lemma 4.18. It is stated separately to demonstrate the proof method in general.

Lemma 4.17. *Suppose $kn < m < (k+1)n$. If $\phi \in \mathcal{S}_k$ and $(y_i - y_{i+1})\phi \in \mathfrak{a}^k$ for all i , then $\phi \in I_c$.*

Proof. We simplify the notation $v_i := v_i^{(c-1)}$. From Theorem 4.2, we know that $I_{c-1}^W \bmod \mathfrak{a}^{k+1}$ is spanned by v_{m-n+1}, \dots, v_{kn} . Therefore

$$\phi \equiv \sum_{\alpha=m-n+1}^{kn} g_\alpha v_\alpha \bmod \mathfrak{a}^{k+1} \quad (33)$$

where $g_\alpha \in \mathcal{H}_c$.

For $\alpha = m - n + 1, \dots, kn$, by definition

$$v_\alpha \equiv \binom{c-1}{k} \sum_{j_1 + \dots + j_k = \alpha} u_{j_1} \cdots u_{j_k} \bmod \mathfrak{a}^{k+1}.$$

We can order $u_{j_1} \cdots u_{j_k}$ so that $j_1 \geq j_2 \geq \dots \geq j_k$ and use the lexicographic order. Under this order, $u_n^{k-1} u_{\alpha-(k-1)n}$ is the maximal term. Since $p_k \equiv k u_k \bmod \mathfrak{a}$, we have that

$$\phi = \left(\frac{g_{m-n+1}}{m-n+1} p_{m-kn+1} + \dots + \frac{g_{kn}}{n} p_n \right) u_n^{k-1} + \text{lower terms in lexicographic order} \bmod \mathfrak{a}^{k+1}.$$

Then modulo \mathfrak{a}^{k+1} , the part in $(y_i - y_{i+1})\phi$ that is divisible by u_n^{k-1} is the product of u_n^{k-1} and

$$(g_{m-n+1}(x_i^{m-kn} - x_{i+1}^{m-kn}) + \cdots + g_{kn-1}(x_i^{n-2} - x_{i+1}^{n-2}) + kg_{kn}(x_i^{n-1} - x_{i+1}^{n-1})). \quad (34)$$

The factor k in the last summand comes from the multiplicity of p_n in p_n^k being k .

Hence $(y_i - y_{i+1})\phi \in \mathfrak{a}^k$ implies that for all i , the polynomial (34) lies in \mathfrak{a} . This means that

$$(g_{m-n+1}, \dots, g_{kn-1}, kg_{kn}) \in \text{Ker}(B_{m-kn, n-1}) \quad \text{modulo } \mathfrak{a}.$$

By Corollary 4.12, there exist $h_j \in \mathcal{H}_c$ such that one has an equality

$$(g_{m-n+1}, \dots, g_{kn-1}, kg_{kn}) = (h_1, \dots, h_n)A_{m-kn, n-1} \quad \text{modulo } \mathfrak{a}.$$

Therefore, by (22) and (33) we have

$$\begin{aligned} \phi &= \sum_j (h_j x_j^{n-1} v_{m-n+1} + \cdots + h_j x_j^{m-n} v_{kn}) - (k-1)v_{kn} \quad \text{modulo } \mathfrak{a}^{k+1}. \\ &= \sum_j h_j (x_j^{n-1} v_{m-n+1} + \cdots + x_j^{m-n} v_{kn}) - (k-1)v_{kn} \quad \text{modulo } \mathfrak{a}^{k+1}. \end{aligned}$$

Combining with equation (23): $f_i \equiv \sum_{j=m-nk}^{n-1} x_i^j v_{m-j} \pmod{\mathfrak{a}^{k+1}}$, we see that

$$\phi = \sum_j h_j f_j - (k-1)g_{kn}v_{kn} \quad \text{modulo } \mathfrak{a}^{k+1}.$$

It remains to show $g_{kn}v_{kn} \in I_c$. Notice that $(k-1)(y_i - y_{i+1})(g_{kn}v_{kn}) = (y_i - y_{i+1})(\phi - \sum h_j f_j) \in \mathfrak{a}^k$ for all i implies that $(0, \dots, 0, g_{kn}) \in \text{Ker}(B_{m-kn, n-1})$. Applying Proposition 4.12, we deduce that there exists $h'_i \in \mathcal{H}_c$ such that $(0, \dots, 0, g_{kn}) = (h'_1, \dots, h'_n)A_{m-kn, n-1}$. Therefore $g_{kn}v_{kn} \equiv \sum h'_i f_i \pmod{\mathfrak{a}^k}$ and $g_{kn}v_{kn} \in I_c$. \square

Proposition 4.18. *Suppose $kn < m < (k+1)n$ and $\ell \geq k$. If $\phi \in \mathcal{S}_\ell$ and $(y_j - y_{j+1})\phi \in \mathfrak{a}^\ell$ for all j , then $\phi \in I_c$.*

Proof. Write $\phi = \sum_{i=m-n+1}^{m-1} \phi_i v_i$ where $\phi_{m-n+1}, \dots, \phi_{nk} \in \mathfrak{a}^{\ell-k}$ and $\phi_{nk+1}, \dots, \phi_{m-1} \in \mathfrak{a}^{\ell-k+1}$. Assume (i_1, \dots, i_ℓ) ($i_1 \geq i_2 \geq \dots \geq i_\ell$) is the largest sequence under the lexicographic order such that $u_{i_1} \cdots u_{i_\ell}$ shows up in ϕ with a nonzero coefficient in \mathcal{H}_c . We will use f_i to kill this nonzero term in ϕ without adding higher order terms and as a result, the lemma will follow from induction on (i_1, \dots, i_ℓ) .

By definition, one has

$$v_\alpha \equiv \binom{c-1}{s} \sum_{j_1 + \cdots + j_s = \alpha} u_{j_1} \cdots u_{j_s} \pmod{\mathfrak{a}^{s+1}}$$

where $s = k$ when $m-n+1 \leq \alpha \leq nk$ and $s = k+1$ when $nk+1 \leq i \leq m-1$. The largest term is either case 1: $u_{\alpha-nk} u_n^k$ when $nk+2 \leq i \leq m-1$; case 2: $u_2 u_{n-1} u_n^{k-1}$ when $i = nk+1$ or case 3: $u_{\alpha-n(k-1)} u_n^{k-1}$ when $m-n+1 \leq i \leq nk$.

Case 1: Suppose $u_{i_1} \cdots u_{i_\ell}$ shows up in $\phi_\alpha v_\alpha$ with nonzero coefficient with $nk+1 < j \leq m-1$. We may assume $u_{i_1} \cdots u_{i_\ell} = u_{i_1} \cdots u_{i_\ell-k-1} u_n^k u_{\alpha-nk}$. Up to some linear rearranging, we may also assume the nonzero coefficient of this term completely comes from $\phi_\alpha v_\alpha$. In other words, the coefficients of $u_{i_1} \cdots u_{i_\ell}$ in $\phi_\gamma v_\gamma$ for $\gamma \neq \alpha$ add up to be 0. Similarly, if the coefficient of $u_{i_1} \cdots u_{i_\ell-k-1} u_n^k u_\beta$ in ϕ is nonzero for $\beta < \alpha - nk$, we may assume the coefficient is completely contributed by $u_{i_1} \cdots u_{i_\ell-k-1} v_i$ for $nk+2 \leq i \leq m-1$ or $u_{i_1} \cdots u_{i_\ell-k-1} u_n v_i$ for $m-n+1 \leq i \leq nk$ (depending on the size of β).

Inside $(y_j - y_{j+1})\phi$, $((y_j - y_{j+1})u_{\alpha-nk})u_{i_1} \cdots u_{i_\ell-k-1} u_n^k$ can only be combined with like terms in the form of $((y_j - y_{j+1})u_\beta)u_{i_1} \cdots u_{i_\ell-k-1} u_n^k$ for $\beta < \alpha - nk$.

Assume the coefficients of $u_{i_1} \cdots u_{i_{\ell-k-1}} u_n v_i$ for $m-n+1 \leq i \leq nk$ and $u_{i_1} \cdots u_{i_{\ell-k-1}} v_i$ for $nk+2 \leq i \leq m-1$ are $h_{m-n+1}, \dots, h_{nk}, h_{nk+2}, \dots, h_{m-1}$ (with some of them being possibly 0). To have $(y_j - y_{j+1})\phi \in \mathfrak{a}^\ell$, by Lemma 4.4, there has to be

$$(z_{nk+2}h_{nk+2}, \dots, z_{m-1}h_{m-1}, 0, z_{m-n+1}h_{m-n+1}, \dots, z_{nk}h_{nk}) \in \text{Ker}(B_{1,n-1}) \pmod{\mathfrak{a}}$$

for some positive integers $z_{m-n+1}, \dots, z_{nk}, z_{nk+2}, \dots, z_{m-1}$ due to possible nontrivial multiplicities (as in (34)).

Since $\text{Ker}(B_{1,n-1}) = \text{Im}(A_{1,n-1})$, by Corollary 4.12 there exists (g_1, \dots, g_n) such that

$$(z_{nk+2}h_{nk+2}, \dots, z_{m-1}h_{m-1}, 0, z_{m-n+1}h_{m-n+1}, \dots, z_{nk}h_{nk}) = (g_1, \dots, g_n)A_{1,n-1} \pmod{\mathfrak{a}}$$

Recall that by Lemma 4.1, $u_n \equiv x_i^n \pmod{\mathfrak{a}}$ while (22) says $f_i = \sum_{j=0}^{n-1} (x_i^j + x_i^{j-2}u_2 + \cdots + u_j)v_{m-j}^{(c-1)}$. Therefore modulo \mathfrak{a}^{k+2} we have

$$\begin{aligned} & z_{nk+1}h_{nk+2}v_{nk+2} + \cdots + z_{m-1}h_{m-1}v_{m-1} + z_{m-n+1}h_{m-n+1}u_n h_{m-n+1} + \cdots + z_{nk}h_{nk}u_n u_{nk} \\ &= \sum_i g_i (x_i^{n-1}v_{nk+2} + \cdots + x_i^{(k+1)n-m+2}v_{m-1} + x_i^{n(k+1)-m}u_n v_{m-n+1} + \cdots + x_i u_n v_{nk}) \\ &\equiv \sum_i g_i x_i^{n(k+1)-m+1} f_i. \end{aligned}$$

Now write $(*) := \phi - \frac{1}{z_\alpha} u_{i_1} \cdots u_{i_{\ell-k+1}} \sum g_i x_i^{n(k+1)-m+1} f_i$. Then $(*)$ still satisfies that $(y_j - y_{j+1})(*) \in \mathfrak{a}^\ell$ for all j and the coefficient of $u_{i_1} \cdots u_{i_{\ell-k-1}} u_n^k u_{\alpha-nk}$ in $(*)$ is 0 with no higher-order summand added compared to ϕ . Our goal is achieved.

Case 2: Suppose the nonzero coefficient of $u_{i_1} \cdots u_{i_\ell}$ in ϕ comes from $\phi_{nk+1}v_{nk+1}$ and so $u_{i_1} \cdots u_{i_\ell} = u_{i_1} \cdots u_{i_{\ell-k-1}} u_2 u_{n-1} u_n^{k-1}$. Similar as case 1, inside $(y_j - y_{j+1})\phi$, terms $((y_j - y_{j+1})u_2)u_{i_1} \cdots u_{i_{\ell-k-1}} u_n^k$ can only be combined with like terms in the form of $((y_j - y_{j+1})u_\beta)u_{i_1} \cdots u_{i_{\ell-k-1}} u_{n-1} u_n^{k-1}$ for $\beta < 2$, which are 0. Set the coefficient of $u_{\ell_1} u_{i_{\ell-k-1}} u_2 u_{n-1} u_n^{k-1}$ to be h_{nk+1} . Then

$$(0, \dots, 0, h_{nk+1}, 0, \dots, 0) \in \text{Ker}(B_{1,n-1}) = \text{Im}(A_{1,n-1}).$$

Here h_{nk+1} lies in the $(n(k+1) - m + 1)$ -th entry. Therefore there exists $g_i \in \mathcal{H}_c$ such that

$$h_{nk+1}v_{nk+1} \equiv \sum g_i (x_i^{n-1}v_{m-n+1} + \cdots + x_i^{m-nk-1}v_{nk+1} + \cdots + x_i v_{m-1}) = \sum g_i f_i \pmod{\mathfrak{a}^{k+2}}.$$

Now the highest degree in $\phi - u_{i_1} \cdots u_{i_{\ell-k-1}} \sum g_i f_i$ is lower than the original (i_1, \dots, i_ℓ) .

Case 3: Suppose the nonzero coefficient of $u_{i_1} \cdots u_{i_\ell}$ in ϕ comes from $\phi_\alpha v_\alpha$ for $m-n+1 \leq \alpha \leq nk$. Then $u_{i_1} \cdots u_{i_\ell} = u_{i_1} \cdots u_{i_{\ell-k}} u_n^{k-1} u_{\alpha-n(k-1)}$. Given case 1, we may assume $i_1, \dots, i_\ell < n$. If the coefficient of $u_{i_1} \cdots u_{i_{\ell-k}} u_n^{k-1} u_\beta$ in ϕ is nonzero, we may assume the coefficient is completely contributed by $u_{i_1} \cdots u_{i_{\ell-k}} v_i$ for $m-n+1 \leq i \leq nk$.

Inside $(y_j - y_{j+1})\phi$, $((y_j - y_{j+1})u_{\alpha-nk})u_{i_1} \cdots u_{i_{\ell-k}} u_n^{k-1}$ can only be combined with like terms in the form of $((y_j - y_{j+1})u_\beta)u_{i_1} \cdots u_{i_{\ell-k}} u_n^{k-1}$ for $\beta < \alpha - nk$. Assume the coefficient of $u_{i_1} \cdots u_{i_{\ell-k}} v_i$ is h_i , $i = m-n+1, \dots, nk$ (with some of the h_i possibly being 0). Then

$$(z_{m-n+1}h_{m-n+1}, \dots, z_{nk}h_{nk}) \in \text{Ker}(B_{m-nk, n-1}) = \text{Im}(A_{m-nk, n-1}).$$

for some positive rational numbers z_{m-n+1}, \dots, z_{nk} due to possible nontrivial multiplicities. Therefore there exists $g_i \in \mathcal{H}_c$ such that

$$z_{m-n+1}h_{m-n+1}h_{m-n+1} + \cdots + z_{nk}h_{nk}u_{nk} = \sum g_i (x_i^{n-1}v_{m-n+1} + \cdots + x_i^{m-n(k-1)+1}v_{nk}) \pmod{\mathfrak{a}^{k+1}},$$

and

$$u_{i_1} \cdots u_{i_{\ell-k}} (z_{m-n+1}h_{m-n+1}u_{m-n+1} + \cdots + z_{nk}h_{nk}u_{nk}) \equiv u_{i_1} \cdots u_{i_{\ell-k}} \sum_i g_i f_i \pmod{\mathfrak{a}^{\ell+1}}.$$

Similar to the previous cases the highest degree in $\phi - \frac{1}{z_\alpha} u_{i_1} \cdots u_{i_{\ell-k}} \sum_i g_i f_i$ is lower than the original (i_1, \dots, i_ℓ) . \square

4.4.1 Proof of the main theorem

With all the puzzle pieces collected, we summarize the proof of Theorem 3.16.

Proof. By Corollary 3.12 and Lemma 3.17, we need to show $F_\ell^\alpha L_c \supset F_\ell^{\text{ind}} L_c$ for all $\ell \geq 0$ and $c = \frac{m}{n} > 1$ with $(m, n) = 1$. We do so by induction on both c using the order in the proof of Lemma 3.13 and F_ℓ .

The base case of F_0 is shown in Lemma 3.18 for any $c > 1$. On the other hand, for all $c > 0$ $F_0^\alpha eL_c = \mathbb{C} \cdot 1 = F_0^{\text{ind}} eL_c$.

For $c > 1$ and $\ell > 0$, assume we have shown that $F_i^\alpha eL_{c'} = F_i^{\text{ind}} eL_{c'}$ for all $i \geq 0$ and c' such that $c' \prec c$, and assume also that $F_{\ell'}^\alpha L_c = F_{\ell'}^{\text{ind}} L_c$ for $0 \leq \ell' < \ell$. Note that because F^α and F^{ind} are both compatible with the isomorphism $eL_{\frac{m}{n}} = eL_{\frac{n}{m}}$ when $n, m > 1$, and that $\frac{n}{m} \prec \frac{m}{n}$ when $n > m$, our assumptions also imply that $F_i^\alpha eL_{c-1} = F_i^{\text{ind}} eL_{c-1}$ for all $i \geq 0$.

Now take a homogeneous $\phi \in (\mathfrak{a}^{\ell+1})^{\perp c}$ such that $0 \neq \overline{\Phi}_c(\phi) \in F_\ell^\alpha L_c$. If $\phi \in \mathcal{S}_\ell^{\perp c}$, by Lemma 3.20, we conclude from the induction hypothesis on c that $\overline{\Phi}_c(\phi) \in F_\ell^{\text{ind}} L_c$.

Hence we may assume $\phi \in \mathcal{S}_\ell$, in which case $\ell \geq \lfloor c \rfloor$. If $\phi \in \mathbb{C}[\mathfrak{h}]^W$, then by Lemma 3.6, $(y_i - y_{i+1})\phi \in (\mathfrak{a}^\ell)^{\perp c}$; if $\phi \notin \mathbb{C}[\mathfrak{h}]^W$, Proposition 4.18 implies that $(y_i - y_{i+1})\phi \in (\mathfrak{a}^\ell)^{\perp c}$. Because $\ell > 0$, $\overline{\Phi}_c(\phi)$ is not a highest weight vector. Now the equality (21) and the inductive hypothesis on F_ℓ implies that $\overline{\Phi}_c(\phi) \in F_\ell^{\text{ind}} L_c$. This concludes the proof of Theorem 3.16. \square

4.5 Description of F^α for $c < 1$

Let $c = \frac{m}{n} < 1$ for positive integer m coprime to n . Inside L_c one can define $\mathcal{H}_c := \mathfrak{a}^{\perp c}$ and it still holds that $L_c = \mathcal{H}_c \cdot eL_c$. In this setting, \mathcal{H}_c is now only a proper submodule of the regular representation of S_n .

Proposition 4.19. *For all $1 \leq k \leq n-1$, $\ell > 0$ and $\overline{\Phi}_c(\phi) \in F_\ell^\alpha L_c$, we have $\overline{\Phi}_c((y_k - y_{k+1})\phi) \in F_{\ell-1}^\alpha L_c$.*

Proof. We first note that when $c < 1$ the equivalence of statements in Proposition 3.23 holds for the whole L_c rather than merely \mathcal{S}_ℓ . Therefore we only need to show that if $\phi \in (\mathfrak{a}^{\ell+1})^{\perp c} \cap \mathfrak{a}^\ell$ satisfies $(y_k - y_{k+1})\phi \in \mathfrak{a}^\ell$ for all $k = 1, \dots, n-1$ then $\phi \in I_c$.

We first prove the case when $\ell = 1$. Recall the ideal $I_c \subset \mathbb{C}[\mathfrak{h}]$ is generated by

$$f_i = \sum_{j=0}^m x_i^j v_{m-j}^{(c)}, \quad i = 1, \dots, n$$

where

$$v_j^{(c)} \equiv cu_j \pmod{\mathfrak{a}}, \quad j = 2, \dots, n$$

Take $\phi := \sum_{i=2}^n h_i p_i \in (\mathfrak{a}^2)^{\perp c}$ where $h_i \in \mathcal{H}_c$. Since the proposition holds when ϕ is symmetric by Lemma 3.6, we may assume that h_2, \dots, h_n are non-constant. By Lemma 4.4, $(y_k - y_{k+1})\phi \in \mathfrak{a}$ for all $k = 1, \dots, n-1$ if and only if $(2h_2, \dots, nh_n) \in \text{Ker}(B_{2,n-1})$. By Corollary 4.12, there exists g_1, \dots, g_n such that $(2h_2, \dots, nh_n) = (g_1, \dots, g_n)A_{2,n-1}$. Therefore,

$$\sum_{i=2}^n h_i p_i \equiv \sum_{i=2}^n i h_i u_i \equiv \sum_{j=1}^n \sum_{i=2}^n g_j x_j^{n+1-i} u_i \pmod{\mathfrak{a}}.$$

Note that $c \sum_{i=2}^m x_j^{n+1-i} u_i \equiv x_j^{n+1-m} (f_j - x_j^m) \pmod{\mathfrak{a}}$. Thus we obtain that

$$\begin{aligned} \sum_{j=1}^n \sum_{i=2}^n g_j x_j^{n+1-i} u_i &\equiv \frac{1}{c} \sum_{j=1}^n g_j \left(x_j^{n+1-m} (f_j - x_j^m) + x_j^{n-m} v_{m+1}^{(c)} + \dots + x_j v_n^{(c)} \right) \pmod{\mathfrak{a}} \\ &\equiv \frac{1}{c} \sum_{j=1}^n g_j \left(x_j^{n+1-m} f_j + x_j^{n-m} v_{m+1}^{(c)} + \dots + x_j v_n^{(c)} \right) \pmod{\mathfrak{a}}. \end{aligned}$$

The second equality follows as $x_j^{n+1} \in \mathfrak{a}$. Since the ideal $I_c^W \subset \mathbb{C}[\mathfrak{h}]^W$ is generated by $v_{m+1}^{(c)}, \dots, v_{m+n-1}^{(c)}$, we conclude that $\sum_{i=2}^n h_i p_i \in I_c$.

When $\ell > 1$, suppose $(y_k - y_{k+1})\phi \in \mathfrak{a}^{\ell+1}$ for all k . Assume (i_1, \dots, i_ℓ) ($i_1 \geq i_2 \geq \dots \geq i_\ell$) is the largest sequence under the lexicographic order such that $u_{i_1} \cdots u_{i_\ell}$ shows up in ϕ with a nonzero coefficient in \mathcal{H}_c . Similar to the proof of Proposition 4.18, we will use f_i to kill this nonzero term in ϕ without adding higher-order terms, and as a result, the proposition will follow from induction on (i_1, \dots, i_ℓ) .

For $\gamma = 2, \dots, i_\ell$, let $h_\gamma \in \mathcal{H}_c$ be the coefficient of $u_{i_1} \cdots u_{i_{\ell-1}} u_\gamma$ in ϕ . Inside $(y_k - y_{k+1})\phi$, $h_{i_\ell}((y_k - y_{k+1})u_{i_\ell})u_{i_1} \cdots u_{i_{\ell-1}}$ can only be combined with like terms in the form of $h_\gamma((y_k - y_{k+1})u_\gamma)u_{i_1} \cdots u_{i_{\ell-1}}$ for $\gamma = 2, \dots, i_\ell - 1$. Therefore $(y_k - y_{k+1})\phi \in \mathfrak{a}^{\ell+1}$ for all k implies that

$$(z_2 h_2, \dots, z_n h_n) \in \text{Ker}(B_{1,n-1}) \pmod{\mathfrak{a}}$$

where z_i 's are integers with $z_{i_\ell} \neq 0$ and $z_\gamma = 0$ for $\gamma > i_\ell$. By the discussion above in the case of $\ell = 1$, we can express $\sum_{\gamma=2}^n z_\gamma h_\gamma u_\gamma$ by f_1, \dots, f_n . Write $(*) := \phi - \frac{1}{z_{i_\ell}} (\sum_{\gamma=2}^n z_\gamma h_\gamma u_\gamma) u_{i_1} \cdots u_{i_{\ell-1}}$. Then $(*)$ still satisfies $(y_k - y_{k+1})(*) \in \mathfrak{a}^\ell$ for all k and the coefficients of $u_{i_1} \cdots u_{i_\ell}$ in $(*)$ is 0 with no higher-order term added compared to ϕ . This concludes the proof of the proposition. \square

A Distributive lattices

One can use the language of distributive lattice to further describe the structure captured by (32) in Proposition 4.11.

Following [BGS88] [Bez95], we define the following notions:

- A set of subspaces of vector space U is called a lattice if it is closed under taking sum and intersections.
- A lattice is called distributive if $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ for any elements U_1, U_2, U_3 in the lattice.
- A n -tuple (U_1, \dots, U_n) of subspaces in U is called distributive if the lattice generated by U_1, \dots, U_n is distributive.
- A n -tuple is called predistributive if all the $(n-1)$ -tuples $(U_1, \dots, \hat{U}_i, \dots, U_n)$, $i = 1, \dots, n$, are distributive.
- A n -tuple is called acyclic if all the 3-triples $(U_1 \cap \cdots \cap U_i, U_{i+1}, U_{i+2} + \cdots + U_n)$, $i = 1, \dots, n-2$, are distributive.

Lemma A.1. [BGS88, Lemma 4.5.2] *A n -tuple (U_1, \dots, U_n) of subspaces in U is distributive if and only if it is both predistributive and acyclic.*

Write $V_i = \{(x_i^{n-1} \phi, \dots, x_i^{m-n} \phi) | \phi \in \mathcal{R}\} \subset \mathcal{R}_n^{2n-m}$ for $i = 1, 2, \dots, n$. We will finally show that

Proposition A.2. *The n -tuple (V_1, \dots, V_n) is distributive.*

To prove this, we first establish the following lemma:

Lemma A.3. *For any $1 \leq j < i$, we have*

$$V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i) = (V_1 \cap \cdots \cap V_j \cap V_{j+1}) + (V_1 \cap \cdots \cap V_j \cap V_{j+2}) + \cdots + (V_1 \cap \cdots \cap V_j \cap V_i)$$

Proof. Consider the short exact sequences

$$V_1 \cap V_2 \cap \cdots \cap V_j \cap V_{j+1} \rightarrow V_1 \cap V_2 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i) \xrightarrow{J_{j+1}} (V_1 \cap V_2 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i)) J_{j+1}$$

$$\begin{aligned} [(V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i)) J_{j+1}] \cap V_{j+2} &\rightarrow (V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i)) J_{j+1} \\ &\xrightarrow{J_{j+2}} (V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i)) J_{j+1} J_{j+2} \\ &\dots \end{aligned}$$

$$[(V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1} \cdots J_{i-2}}] \cap V_i \rightarrow (V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1} \cdots J_{i-2}} \\ \xrightarrow{J_{i-1}} (V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1} \cdots J_{i-1}},$$

from which we obtain

$$\dim V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i) = \dim V_1 \cap V_2 \cap \cdots \cap V_j \cap V_{j+1} \\ + \dim[(V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1}}] \cap V_{j+2} + \cdots \\ + \dim(V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1} \cdots J_{i-1}}.$$

On the other hand, for $1 \leq j < i$ and $1 \leq k \leq i - j$, there are inclusions

$$[(V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i))^{J_{j+1} \cdots J_{j+k-1}}] \cap V_{j+k} \\ \subset [V_1 \cap \cdots \cap V_j \cap \text{Ker}(J_{j+1} \cdots J_{j+k})]^{J_{j+1} \cdots J_{j+k-1}} \subset V_1 \cap \cdots \cap V_j \cap V_{j+k}.$$

As a consequence

$$\dim V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i) \leq \sum_{k=1}^{i-j} \dim V_1 \cap \cdots \cap V_j \cap V_{j+k}.$$

Therefore the inclusion

$$V_1 \cap \cdots \cap V_j \cap (V_{j+1} + \cdots + V_i) \supset (V_1 \cap \cdots \cap V_j \cap V_{j+1}) + (V_1 \cap \cdots \cap V_j \cap V_{j+2}) + \cdots + (V_1 \cap \cdots \cap V_j \cap V_i)$$

has to be an equality. \square

Now we are ready to prove Proposition [A.2](#).

Proof of Proposition [A.2](#). We show the lemma by induction. The cases of $n = 1$ or 2 is trivial. When $n = 3$, a 3-tuple (U_1, U_2, U_3) is distributive if and only if $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ ([BGS88](#), example after Lemma 4.5.1). Our (V_1, V_2, V_3) satisfies this property.

Now assume any $(n - 1)$ -subtuple of (V_1, \dots, V_n) is distributive. In other words, (V_1, \dots, V_n) is pre-distributive. By Lemma [A.1](#), it only remains to show that (V_1, \dots, V_n) is acyclic, i.e. all the 3-triples $(V_1 \cap \cdots \cap V_i, V_{i+1}, V_{i+2} + \cdots + V_n)$, $i = 1, \dots, n - 2$, are distributive. But this follows from the identity

$$V_1 \cap \cdots \cap V_i \cap (V_{i+1} + (V_{i+2} + \cdots + V_n)) \\ = (V_1 \cap \cdots \cap V_i \cap V_{i+1}) + (V_1 \cap \cdots \cap V_i \cap V_{i+2}) + \cdots + (V_1 \cap \cdots \cap V_i \cap V_n) \\ = (V_1 \cap \cdots \cap V_i \cap V_{i+1}) + (V_1 \cap \cdots \cap V_i \cap (V_{i+2} + \cdots + V_n))$$

using Lemma [A.3](#) twice. \square

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