θ-TRIANGLE AND ω-PARALLELOGRAM PAIRS WITH COMMON AREA AND COMMON PERIMETER

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ABSTRACT. We show that given a convex angle θ, there exist, except for finitely many exceptions, infinitely many pairs of integral θ-triangle and ω-parallelogram with common area and common perimeter satisfying that $\sin \omega$ is a previously fixed rational multiple of $\sin \theta$. This is achieved by relating the problem to a family of elliptic curves. We also study the elliptic surfaces and the elliptic threefold that are obtained by varying one or two parameters.

1. INTRODUCTION

Geometry and Number Theory are closely connected. The simplest examples may be Diophantine equations coming from plane geometry such as the ancient congruent number problem and the Pythagorean theorem that inspired the raise of Fermat's last theorem. As simple as right triangles and squares are, there are many interesting questions about them, which yield unexpected relations and methods.

The congruent number problem asks about the existence of a right triangle with rational side lengths and whose area is equal to a given natural number. This question has been generalized in several directions. Fujiwara [Fuj98] proposed to extend this problem to other angles. A natural number is θ-congruent if there exists a rational triangle of area $n\sqrt{r^2 - s^2}$ with an angle equal to θ, where $\cos \theta = \frac{s}{r}$ is a rational number. Such a triangle is called a θ-triangle. The cases where $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ are of particular interest, since, together with $\frac{\pi}{2}$, they correspond to rational multiples of π with rational cosine.

These types of connections extend to other geometric objects. To answer a question of Sands, Guy [Guy95] found that there are no pairs of an integral right triangle and a rectangle with the same area and the same perimeter. Instead, he proved that there are infinitely many such pairs involving an integral isosceles triangle and an integral rectangle. Since then, several variations of this problem have been studied. Bremner and Guy [BG06] generalized this statement to pairs involving a Heron triangle (a triangle with rational sides and area) and a rectangle. Zhang and Peng [ZP17] showed the analogous problem for Heron triangle and parallelogram pairs. The case of right triangle and parallelogram pairs was considered by Zhang [Zha16], while Chern [Che16] studied integral right triangle and θ-integral rhombus pairs. Das, Juyal, and Moody [DJM17] showed the analogous statement for integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs. Zhang, Peng, and Wang studied several questions along the same lines involving trapezoids [ZPJ18]. Recently Hirakawa and Matsumura [HM19] proved that there is a unique pair of a rational right triangle and a rational isosceles triangle with the same perimeter and the same area. Further questions in this direction involving pairs of Heron triangles were considered by Kramer and Luca [KL00], Bremner [Bre06], and van Luijk [vL07]. Questions involving pairs of other types of triangles were considered by Choudhry [Cho07].

It is natural to ask how we can generalize questions such as the search for pairs of right triangles and parallelograms to the θ-triangle case, when $\cos \theta$ is rational. This is the aim of this paper.

Suppose that a triangle has sides of length x, y, z and that the angle between the sides of length x and y equals θ. Let $\cos \theta = a = \frac{s}{r}$, $r, s \in \mathbb{Z}$, $|s| < r$, $(s, r) = 1$.

We look for pairs of θ-triangles and ω-parallelograms with rational sides, equal perimeter, and equal area. In this article we only consider pairs up to similarity.

The law of cosines implies $z^2 = x^2 + y^2 - 2axy$. We also have $|a| < 1$. By applying the standard parametrization technique for a quadric, we can write

$$(x, y, z) = (n^2 - m^2, 2m(n - am), m^2 - 2amn + n^2),$$

where n, m are rationals (or integers, if we allow scaling) satisfying conditions so that the quantities above are positive.

Let the sides of the corresponding parallelogram be p, q, and let the intersection angle between them be ω, where $0 < \omega \leq \frac{\pi}{2}$. We thus have

(1)
$$\begin{cases} \sqrt{1 - a^2}(n^2 - m^2)m(n - am) = pq \sin \omega, \\ \\ n^2 + (1 - a)mn - am^2 = p + q. \end{cases}$$

Since m, n, p, q are rational, we must have $\sin \omega = t\sqrt{1 - a^2}$, with t a rational number.

With this change of variables system (1) becomes

(2)
$$\begin{cases} (n^2 - m^2)m(n - am) = tpq, \\ \\ (n - am)(n + m) = p + q. \end{cases}$$

Our main result is the following.

Theorem 1. *Given θ with $\cos \theta = a \in \mathbb{Q}$, for all but finitely many $t \in \mathbb{Q}$ with $0 < t \leq \frac{1}{\sqrt{1-a^2}}$, there exist infinitely many pairs of integral θ-triangle and integral ω-parallelogram such that $\sin \omega / \sin \theta = t$ with common area and common perimeter.*

A common restriction in previous works [DJM17, Zha16] is to request that $\cos \omega$ be also rational.

Theorem 2. *Given θ with $\cos \theta = a \in \mathbb{Q}$, for all but finitely many $\tau \in \mathbb{Q}$ with $0 < \tau$, $a\tau < 1$, and $2a\tau < \tau^2 + 1$, there exist infinitely many pairs of integral θ-triangle and integral ω-parallelogram such that $\sin(\omega/2)/\sin(\omega/2 + \theta) = \tau$ with common area and common perimeter.*

While the consideration of $\sin(\omega/2)/\sin(\omega/2+\theta)$ seems rather technical, it is a natural geometric condition that guarantees that both $\sin \omega$ and $\cos \omega$ are rational multiples of $\sin \theta$ and $\cos \theta$ respectively. When $\theta = \frac{\pi}{2}$ this reduces to $\tan(\omega/2) = \tau$.

The results are obtained by reformulating the problems in terms of families of elliptic curves and proving that the elliptic curves have positive rank for all but finitely many t (or τ).

The paper is organized as follows. In Section 2 we prove Theorem 1 by relating it to an elliptic curve on the parameters a, t. We consider some particular cases of Theorem 1 in Section 3 by fixing the values of one of the parameters a, t or the relation between them. In Section 4 we study the structure of the Mordell–Weil group: we show that all the possible torsion groups from Mazur's Theorem with a subgroup of order 2 are realized for some specific values of a, t in Subsection 4.1, and we compute the rank and characterize a generator in Subsection 4.2 by studying the corresponding rational elliptic surface resulting from fixing the value of a but letting t free as a parameter. By Silverman's Specialization Theorem, this gives a lower bound for the rank of almost all fibers. Theorem 2 is proven in Section 5 by relating to a different elliptic curve on parameters a, τ. Sections 6 and 7 treat the ranks over $\mathbb{C}(\tau)$ and $\mathbb{Q}(\tau)$ respectively by considering the elliptic $K3$-surface resulting from leaving τ as a variable and the elliptic threefold that is given when both a and τ are kept as variables.

2. Proof of Theorem 1

In this section we prove Theorem 1 by combining several auxiliary results.

Proposition 3. *Let $a, t \in \mathbb{Q}$, $|a| < 1$, $0 < t \leq \frac{1}{\sqrt{1-a^2}}$. If the elliptic curve*

(3)
$$E_{a,t} : Y^2 = X^3 + t((1 + a)^2 t + 4(a - 3))X^2 + 32(1 - a)t^2 X$$

has infinitely many points on $E_{a,t}(\mathbb{Q})$ satisfying the conditions

(4)
$$4(1 - a)t < X < 8t, \qquad |Y| < (1 + a)tX,$$

then there exist infinitely many pairs of integral θ-triangle and integral ω-parallelogram with common area and common perimeter such that $\cos\theta = a$ and $\sin\omega/\sin\theta = t$.

Proof. From system (2) we have

$$(n - am)^2(n + m)^2 - \frac{4}{t}(n^2 - m^2)m(n - am) = (p + q)^2 - 4pq = (p - q)^2.$$

By considering the change of variables

$$\begin{cases} Y = 4(1 - a^2)t^2 \frac{p-q}{(n-am)^2}, \\ X = 4(1 - a)t\frac{n+m}{n-am}. \end{cases}$$

we obtain the Weierstrass form (3).

Given a point $(X, Y) \in E_{a,t}(\mathbb{Q})$, we can take

$$\begin{cases} m = X - 4(1 - a)t, \\ n = aX + 4(1 - a)t, \end{cases} \qquad \begin{cases} p = 2(1 - a^2)((1 + a)tX + Y), \\ q = 2(1 - a^2)((1 + a)tX - Y), \end{cases}$$

and the sides of the triangle adjacent to θ to be

$$\begin{cases} x = (1 - a^2)X(8t - X), \\ y = 8(1 - a^2)t(X - 4(1 - a)t). \end{cases}$$

For the above parameters to satisfy the conditions of our problem we need $x, y, p, q > 0$. Since $|a| < 1$ and $0 < t$, it suffices to have

$$X(8t - X) > 0, \qquad X - 4(1 - a)t > 0, \qquad (1 + a)tX \pm Y > 0.$$

The above inequalities can be condensed as conditions (4) of the statement. From the expressions for m, n, p, q, it is easy to see that any point $(X, Y) \in E_{a,t}(\mathbb{Q})$ satisfying (4) leads to a distinctive pair. $\square$

Remark 4. *Let us show how to construct ω from t. To this end, construct a triangle ABC with angle θ at A, $|AC| = t$, $|BC| = 1$, and the angle at B acute (see Figure 1). We denote this angle by α.*
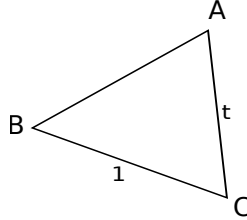

FIGURE 1. Construction of ω from a given t.

This construction is possible precisely when $\frac{1}{t} > \sin\theta = \sqrt{1 - a^2}$, which is the condition that we have. Then the law of sines implies

$$\frac{1}{\sin\theta} = \frac{|BC|}{\sin\theta} = \frac{|AC|}{\sin\alpha} = \frac{t}{\sin\alpha},$$

which implies $\sin\alpha = t\sin\theta$ and $\alpha = \omega$.

To prove Theorem 1 it remains to prove that, given a, for all but finitely many t, there are infinitely many points on $E_{a,t}(\mathbb{Q})$ satisfying the conditions (4).

Lemma 5. *For given $a \in \mathbb{Q}$ with $|a| \neq 1$, the point*

$$P_{a,t} = (8t, 8(1 + a)t^2)$$

has infinite order in $E_{a,t}$ for all but finitely many t. In particular, $E_{a,t}(\mathbb{Q})$ has positive rank for all but finitely $t \in \mathbb{Q}$.

Proof. First we record that the discriminant of $E_{a,t}$ is given by

$$\Delta_{a,t} = 2^{14}(1-a)^2(1+a)^2 t^6((1+a)^2 t^2 + 8(a-3)t + 16).$$

Therefore, we have an elliptic curve as long as

$$a \neq \pm 1 \text{ and } t \neq \frac{4(3-a) \pm 8\sqrt{2(1-a)}}{(1+a)^2}.$$

The point $P_{a,t} = (8t, 8(1+a)t^2)$ can be found by numerical experimentation. We also obtain the point $(0,0)$ of order 2. In Subsection 4.2 we will use the theory of elliptic surfaces to determine that, when we keep t as a variable, the rank of $E_a(\mathbb{Q}(t))$ is 1 and that $P_a(t)$ is a generating point.

If we only need to prove that $P_{a,t}$ is a point of infinite order, it suffices to consider $kP_{a,t}$ and $kP_{a,t} + (0,0)$ for $k = 1, \ldots, 4$. For the record, they are as follows.

$$
\begin{aligned}
P_{a,t} =& (8t, 8(1+a)t^2) \\
P_{a,t} + (0,0) =& (4(1-a)t, -4(1-a^2)t^2) \\
2P_{a,t} =& (4, -4((a-3)t+2)) \\
2P_{a,t} + (0,0) =& (8(1-a)t^2, 8(1-a)((a-3)t+2)t^2) \\
3P_{a,t} =& \left(\frac{8t((a-1)t+1)^2}{(2t-1)^2}, -\frac{8(a+1)t^2((a-1)t+1)(2(a-1)t^2+(a-3)t+3)}{(2t-1)^3} \right) \\
3P_{a,t} + (0,0) =& \left(-\frac{4(a-1)t(2t-1)^2}{((a-1)t+1)^2}, -\frac{4(a^2-1)t^2(2t-1)(2(a-1)t^2+(a-3)t+3)}{((a-1)t+1)^3} \right) \\
4P_{a,t} =& \left(\frac{4(2(a-1)t^2+1)^2}{((a-3)t+2)^2}, \frac{4(2(a-1)t^2+1)}{((a-3)t+2)^3} \right. \\
& \left. \times \left[2(a-1)(a^2-2a+5)t^4 + 8a^2(a-3)(a-1)t^3 - (a^2-22a+25)t^2 - 4(a-3)t - 2 \right] \right) \\
4P_{a,t} + (0,0) =& \left(-\frac{8(a-1)t^2((a-3)t+2)^2}{(2(a-1)t^2+1)^2}, \frac{8(a-1)t^2((a-3)t+2)}{(2(a-1)t^2+1)^3} \right. \\
& \left. \times \left[2(a-1)(a^2-2a+5)t^4 + 8a^2(a-3)(a-1)t^3 - (a^2-22a+25)t^2 - 4(a-3)t - 2 \right] \right)
\end{aligned}
$$

The points above, together with all their negatives, and $(0,0)$, yield 17 points that are generically different. That is, for a fixed value of a, there are only finitely many t's such that some of the points in the list coincide. Mazur's Theorem on the torsion of elliptic curves over the rational numbers [Maz77, Maz78] implies that the torsion subgroup of $E_{a,t}(\mathbb{Q})$ has at most 16 elements. Hence, one of the 17 points is not torsion. In conclusion, for given a, $P_{a,t}$ has infinite order for all but finitely many t. $\square$

In order to finish the proof of Theorem 1, we must find infinitely many points on $E_{a,t}(\mathbb{Q})$ satisfying the conditions (4). To do this, we need the following result.

Theorem 6 (Poincaré and Hurwitz ([Sko50]p.78)). *Let E be an elliptic curve over $\mathbb{Q}$ with positive rank. Then for any rational point $P \in E(\mathbb{Q})$ and any neighborhood $P \in U \subset \mathbb{RP}^2$, one can find infinitely many rational points $P_i \in E(\mathbb{Q})$ such that $P_i \in U$.*

We have now all the elements to prove our main result.

Proof of Theorem 1. If $E_{a,t}(\mathbb{Q})$ contains only one point of order 2, then $P_{a,t}$ is dense in $E_{a,t}(\mathbb{R})$. Otherwise, $E_{a,t}(\mathbb{Q})$ contains three points of order 2 and $E_{a,t}(\mathbb{Q})$ is dense in the connected component where $P_{a,t}$ lies in.

Take δ such that $\frac{1-a}{2} < \delta < 1$ and set $X_0 = \delta 8t$. We will find Y_0 such that $(X_0, Y_0) \in E_{a,t}(\mathbb{R})$.

$$\frac{Y_0^2}{X_0^2} = X_0 + t((1+a)^2 t + 4(a-3)) + \frac{32(1-a)t^2}{X_0}$$
$$= \frac{t}{\delta} 8\left(\delta - \frac{1-a}{2}\right)(\delta - 1) + (1+a)^2 t^2$$
$$< (1+a)^2 t^2.$$

Restricting to δ close enough to 1, we can guarantee that $\frac{Y_0^2}{X_0^2} > 0$ so $(X_0, Y_0) \in E_{a,t}(\mathbb{R})$ satisfies the conditions (4) and lies in the same connected component as $P_{a,t}$. By continuity and by Theorem 6 there are infinitely many points in $E_{a,t}(\mathbb{Q})$ satisfying the conditions (4). $\qquad\square$

3. Particular cases for Theorem 1

In this section we consider some cases of particular interest because of their geometric interpretation. Because of Theorem 1, it is natural to examine the exceptional values for which the point $P_{a,t}$ has finite order.

We recall that the parameters a, t are rational numbers satisfying $|a| < 1$, $0 < t \le \frac{1}{\sqrt{1-a^2}}$.

Case $a = 0$. This corresponds to a right triangle. We have

$$E_{0,t} : Y^2 = X^3 + t(t-12)X^2 + 32t^2 X,$$

which is nonsingular for $t \ne 0$.

Assuming $t \ne 0$, the point $P_{0,t} = (8t, 8t^2)$ has infinite order for $t \ne \frac{1}{2}, \frac{2}{3}, 1$. This includes Zhang's case [Zha16].

The following table summarizes the results for the values of t in which $P_{0,t}$ has finite order. The last column indicates the points in $E_{0,t}(\mathbb{Q})$ leading to solutions (if any), and the corresponding solutions up to geometric similarity and excluding symmetric solutions.

t	ω	$E_{0,t}(\mathbb{Q})$	generators	solutions
$\frac{1}{2}$	$\frac{\pi}{6}$	$\mathbb{Z}/6\mathbb{Z}$	$\langle (2,1) \rangle$, $P_{0,\frac{1}{2}} = 2(2,1)$	none
$\frac{2}{3}$	$\sin^{-1}(2/3)$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle P_{0,\frac{2}{3}}, (0,0) \rangle$	$(4,0)$, $\left(\frac{32}{9}, 0\right)$ $p = q = x$, $y = \frac{4}{3}x$
1	$\frac{\pi}{2}$	$\mathbb{Z}/6\mathbb{Z}$	$\langle P_{0,1} \rangle$	none

We remark that the last case when $t = 1$ is the one considered by Bremner and Guy [BG06]. The word "none" indicates that there are only degenerate solutions.

Since $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are the only two angles (besides $\frac{\pi}{2}$) which are rational multiples of π with rational cosine, we discuss these two cases in particular.

Case $a = \frac{1}{2}$. This corresponds to a $\frac{\pi}{3}$-triangle. The curve is given by

$$E_{\frac{1}{2},t} : Y^2 = X^3 + \frac{t}{4}(9t - 40)X^2 + 16t^2 X,$$

which is nonsingular for $t \ne 0, \frac{8}{9}, 8$.

Assuming that $t \ne 0, \frac{8}{9}, 8$, the point $P_{\frac{1}{2},t} = (8t, 12t^2)$ has infinite order for $t \ne -1, \frac{1}{2}, \frac{4}{5}, 1, 2$. The values $t = -1, 2$ are irrelevant for our geometric problem.

The following table summarizes the results in this case.

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t	ω	$E_{\frac{1}{2},t}(\mathbb{Q})$	generators	solutions
$\frac{1}{2}$	$\sin^{-1}(\sqrt{3}/4)$	$\mathbb{Z}/6\mathbb{Z}$	$\langle(1,\frac{3}{4})\rangle$, $P_{\frac{1}{2},\frac{1}{2}} = 2\left(1,\frac{3}{4}\right)$	none
$\frac{4}{5}$	$\sin^{-1}(2\sqrt{3}/5)$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle P_{\frac{1}{2},\frac{4}{5}}, (0,0)\rangle$	$(4,0)$, $\left(\frac{64}{25},0\right)$ $p = q = x,\ y = \frac{8}{5}x$
1	$\frac{\pi}{3}$	$\mathbb{Z}/8\mathbb{Z}$	$\langle P_{\frac{1}{2},1}\rangle$	$(4,\pm2)$ $p = x = y,\ q = p/2$

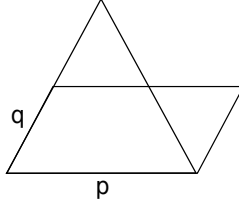The last case corresponds to the equilateral triangle (see Figure 2).

FIGURE 2. The case $a = \frac{1}{2}$, $t = 1$ corresponds to the equilateral triangle.

Case $a = -\frac{1}{2}$. This corresponds to a $\frac{2\pi}{3}$-triangle. We obtain

$$E_{-\frac{1}{2},t} : Y^2 = X^3 + \frac{t}{4}(t - 56)X^2 + 48t^2 X,$$

which is nonsingular for $t \neq 0$.

Assuming that $t \neq 0$, $P_{-\frac{1}{2},t} = \left(8t, 4t^2\right)$ has infinite order for $t \neq \frac{1}{2}, \frac{4}{7}, \frac{2}{3}$.

The following table summarizes the results in this case.

t	ω	$E_{-\frac{1}{2},t}(\mathbb{Q})$	generators (if torsion) or point of infinite order (if positive rank)	solutions
$\frac{1}{2}$	$\sin^{-1}(\sqrt{3}/4)$	$\mathbb{Z}/6\mathbb{Z}$	$\langle(3,\frac{3}{4})\rangle$, $P_{-\frac{1}{2},\frac{1}{2}} = 2\left(3,\frac{3}{4}\right)$	none
$\frac{4}{7}$	$\sin^{-1}(2\sqrt{3}/7)$	$\mathrm{rk}(E_{-\frac{1}{2},\frac{4}{7}}(\mathbb{Q})) > 0$	$\left(\frac{12}{7}, \frac{144}{49}\right)$	$3\left(\frac{12}{7}, \frac{144}{49}\right)$ infinitely many
$\frac{2}{3}$	$\sin^{-1}(1/\sqrt{3})$	$\mathbb{Z}/6\mathbb{Z}$	$\langle P_{-\frac{1}{2},\frac{2}{3}}\rangle$	none

In the case of $t = \frac{4}{7}$ we found a point of infinite order unrelated to $P_{-\frac{1}{2},\frac{4}{7}}$ and by Theorem 1 we obtain infinitely many solutions. We have indicated that the triple of this point satisfies the conditions of Theorem 1.

Now we consider different types of conditions by either imposing values on t or relationships between a and t.

Case $t = 1$. We have $\sin\omega = \sin\theta$. The curve under consideration is

$$E_{a,1} : Y^2 = X^3 + (a^2 + 6a - 11)X^2 + 32(1 - a)X,$$

which is nonsingular since $a \neq \pm1$.

Assuming that $a \neq \pm1$, the point $P_{a,1} = (8, 8(1 + a))$ has finite order for $a = 0, \frac{1}{2}, \frac{2}{3}$.

The following table summarizes the results in this case.

a	ω	$E_{a,1}(\mathbb{Q})$	generators	solutions
0	$\frac{\pi}{2}$	$\mathbb{Z}/6\mathbb{Z}$	$\langle P_{0,1} \rangle$	none
$\frac{1}{2}$	$\frac{\pi}{3}$	$\mathbb{Z}/8\mathbb{Z}$	$\langle P_{\frac{1}{2},1} \rangle$	$(4, \pm 2)$ $p = x = y,\ q = p/2$
$\frac{2}{3}$	$\sin^{-1}(\sqrt{5}/3)$	$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle P_{\frac{2}{3},1}, (0,0) \rangle$	$(3,0),\ \left(\frac{32}{9},0\right)$ $y = \frac{8}{9}x,\ p = q = \frac{2}{3}x$ $\left(4, \pm\frac{4}{3}\right),\ \left(\frac{8}{3}, \pm\frac{8}{9}\right)$ $y = \frac{4}{3}x,\ p = x,\ q = \frac{2}{3}p$

We remark that the cases $a = 0, \frac{1}{2}$ were considered earlier.

Case $a > 0, t = 2a$. This implies $\sin\omega = \sin(2\theta)$. The curve is given by

$$E_{a,2a} : Y^2 = X^3 + 4a(a-1)(a^2 + 3a + 6)X^2 + 128a^2(1-a)X,$$

which is nonsingular since $a \neq 0, \pm 1$.

Assuming that $a \neq 0, \pm 1$, the point $P_{a,2a} = (16a, 32a^2(1+a))$ has infinite order for $a \neq \frac{1}{4}, \frac{1}{2}$.

The case $a = \frac{1}{2}$ was considered earlier.

When $a = \frac{1}{4}$, we have $2\theta = \omega = \sin^{-1}(\sqrt{15}/8)$. $E_{\frac{1}{4},\frac{1}{2}}(\mathbb{Q})$ has positive rank, with a point $(24, 105)$ of infinite order, unrelated to $P_{\frac{1}{4},\frac{1}{2}}$. This point leads to infinitely many solutions. We remark that $4(24, 105)$ satisfies the conditions of Theorem 1.

Case $a < 0, t = -2a$. This happens when $\sin\omega = -\sin(2\theta)$. The curve is given by

$$E_{a,-2a} : Y^2 = X^3 + 4a(a+3)(a^2 - a + 2)X^2 + 128a^2(1-a)X,$$

which is nonsingular since $a \neq 0, \pm 1$.

Assuming that $a \neq 0, \pm 1$, the point $P_{a,-2a} = (-16a, 32a^2(1+a))$ has infinite order for $a \neq -\frac{1}{4}, \frac{1}{2}$. We discard the value $a = \frac{1}{2}$ because we require that $a < 0$ as $0 < t$.

When $a = -\frac{1}{4}$, we have $\omega = \sin^{-1}(\sqrt{15}/8)$ and $\theta = \frac{\pi - \omega}{2}$. Also $E_{-\frac{1}{4},\frac{1}{2}}(\mathbb{Q}) \cong \mathbb{Z}/6\mathbb{Z}$, generated by $\left(\frac{5}{2}, \frac{15}{16}\right)$, where $2\left(\frac{5}{2}, \frac{15}{16}\right) = P_{-\frac{1}{4},\frac{1}{2}}$. This only leads to degenerate solutions.

4. The group of rational points of $E_{a,t}$

In this section we consider the group $E_{a,t}(\mathbb{Q})$ in generality. First we consider the torsion group and show that all the possible groups *with a subgroup of order 2* are realized as torsion groups for certain values of $a, t \in \mathbb{Q}$. Then we consider the free part and show that the elliptic surface $E_a(\mathbb{Q}(t))$ (where a is a parameter taking a specific value and t is kept as a variable) has rank 1 and that $P_a(t)$ is a generator. We do this by considering $E_a(\mathbb{C}(t))$. By a theorem of Silverman, this implies that $E_{a,t}(\mathbb{Q})$ has rank at least 1 for all but finitely many values of t.

4.1. **The torsion group.** By Mazur's Theorem, there are 15 possibilities for the torsion group of an elliptic curve over $\mathbb{Q}$. Since $E_{a,t}$ has always the point $(0,0)$ of order two, this leaves 10 possibilities. By the work from Section 3, we know that the groups $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are all realized as $E_{a,t}(\mathbb{Q})_{\text{tor}}$ for particular choices of $a, t \in \mathbb{Q}$.

It remains to see the cases $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/10\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We will see that these groups all appear as $E_{a,t}(\mathbb{Q})_{\text{tor}}$.

A method to find these cases consists of imposing different conditions for the order of $P_{a,t}$. For example, we get that $3P_{a,t} = O$ iff $t = \frac{1}{2}$, $4P_{a,t} = O$ iff $t = \frac{2}{3-a}$, etc. Some conditions are impossible. For example, we can not find $a, t \in \mathbb{Q}$ such that $P_{a,t} = O, (0,0)$ or $2P_{a,t} = (0,0)$. This implies that we can not have $E_{a,t}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as a whole group (with rank zero). For the other cases, however, we can exhibit examples with rank zero.

We have gathered the examples in the following table.

(a,t)	$E_{a,t}(\mathbb{Q})_{\text{tor}}$	torsion generators	$\text{rk}(E_{a,t}(\mathbb{Q}))$
$\left(0, \frac{1}{3}\right)$	$\mathbb{Z}/2\mathbb{Z}$	$\langle(0,0)\rangle$	> 0
$\left(\frac{47}{72}, 1\right)$	$\mathbb{Z}/4\mathbb{Z}$	$\left\langle\left(\frac{10}{3}, \frac{35}{108}\right)\right\rangle$	> 0
$\left(\frac{1}{4}, \frac{2}{9}\right)$	$\mathbb{Z}/10\mathbb{Z}$	$\langle P_{\frac{1}{4},\frac{2}{9}} + (0,0)\rangle$	0
$\left(\frac{7}{8}, \frac{4}{3}\right)$	$\mathbb{Z}/12\mathbb{Z}$	$\langle P_{\frac{7}{8},\frac{4}{3}}\rangle$	0
$\left(\frac{11}{21}, \frac{7}{8}\right)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle(3,0),(0,0)\rangle$	> 0
$\left(\frac{17}{18}, 3\right)$	$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\langle P_{\frac{17}{18},3}, \left(-\frac{9}{4}, 0\right)\rangle$	0

4.2. **The structure of $E_a(\mathbb{C}(t))$.** In this section we fix the value of a and think of E_a as an elliptic surface over $\mathbb{Q}(t)$. Thus consider the following elliptic surface over $\mathbb{Q}(t)$,

$$E_a : Y^2 = X^3 + t((1+a)^2 t + 4(a-3))X^2 + 32(1-a)t^2 X,$$

with discriminant

$$\Delta_a = 2^{14}(1-a)^2(1+a)^2 t^6 ((1+a)^2 t^2 + 8(a-3)t + 16)$$

and a point

$$P_a = (8t, 8(1+a)t^2),$$

which is generically of infinite order by Lemma 5.

Recall that $|a| < 1$ and $a \in \mathbb{Q}$, so that $\Delta_a \neq 0$ and also its quadratic factor has two different roots $t = \sigma_\pm$ over $\mathbb{C}$.

Theorem 7. *Let a be such that $\Delta_a \neq 0$. The Mordell–Weil rank of the elliptic surface E_a over $\mathbb{C}(t)$ is 1 and P_a is a generator. In addition, $E_a(\mathbb{C}(t))_{\text{tor}} = \langle(0,0)\rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since the coefficients a_2 and a_4 of the Weierstrass form E_a are polynomials of degree two in t, E_a is a rational elliptic surface (see [SS10], 4.10 or [SS], 5.13). Moreover, the Euler characteristic is $\chi(E_a) = 1$ ([SS10], 8.3) and the rank and the discriminant of the Néron–Severi lattice are $\rho(E_a) = 10$ and disc $\text{NS}(E_a) = -1$ ([SS10], 8.8).

The singularities of E_a are at $t = 0$, $t = \infty$, and at the roots $\sigma_\pm$ of $(a+1)^2 t^2 + 8(a-3)t + 16$. By applying Tate's algorithm ([Sil94], IV.9) the singularity at $t = 0$ has type I_0^* in the Kodaira classification, the singularity at $t = \infty$ has type I_4, and the singularities at $t = \sigma_\pm$ have type I_1. By the Shioda–Tate formula ([Shi72], Corollary 1.5 or [SS10], Corollary 6.13), we have

(5) $$\rho(E_a) = \text{rk}\,E_a(\mathbb{C}(t)) + 2 + \sum(m_v - 1),$$

where m_v is the number of components of the corresponding singular fiber. We have $m_v = n$ if the type is I_n and $m_v = 5$ if the type is I_0^*. In our case,

$$10 = \rho(E_a) = \text{rk}\,E_a(\mathbb{C}(t)) + 2 + (5-1) + (4-1) + 2 \cdot (1-1) = \text{rk}\,E_a(\mathbb{C}(t)) + 9.$$

We then conclude that $\text{rk}\,E_a(\mathbb{C}(t)) = 1$.

Recall that $(0,0)$ is a torsion point of order two. The points of order 2 besides $(0,0)$ satisfy

$$X = \frac{t}{2}\left(-((1+a)^2 t + 4(a-3)) \pm (a+1)\sqrt{((a+1)t+4)^2 - 32t}\right).$$

We see that there are no other generic points over $\mathbb{C}(t)$ of order 2.

By Table (4.5) in p. 264 of [MP89], $E_a(\mathbb{C}(t))_{\text{tor}}$ is isomorphic to either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ (because in our case the rank of the Mordell–Weil group is $R = 1$ and the Euler characteristic is $\chi = 1$).

Let $P = (x,y)$ be such that $2P = (0,0)$. Then we have

$$x(2P) = 0 = \frac{x^4 - 2^6(1-a)t^2 x^2 + 2^{10}(1-a)^2 t^4}{y^2} = \left(\frac{x^2 - 2^5(1-a)t^2}{y}\right)^2.$$

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By setting $a = 1 - 2k^2$, we get that $x = \pm 8kt$. Hence $y^2 = 256(1 \mp k)^2 k^2 t^3 \left((1 \pm k)^2 t - 2\right)$ which is not a square in $\mathbb{C}(t)$. Therefore we get that $E_a(\mathbb{C}(t))_{\mathrm{tor}} = \mathbb{Z}/2\mathbb{Z}$.

By formula (22) in 11.10 of [SS10], we have

$$(6) \qquad |\mathrm{disc}\,\mathrm{NS}(E_a)| = \frac{|\mathrm{disc}\,T(E_a) \cdot \mathrm{disc}\,\mathrm{MWL}(E_a)|}{|E_a(\mathbb{C}(t))_{\mathrm{tor}}|^2},$$

where $\mathrm{MWL}(E_a)$ is the Mordell–Weil lattice and $T(E_a)$ is the trivial lattice.

Since $\mathrm{disc}\,\mathrm{NS}(E_a) = -1$, the rank of $\mathrm{MWL}(E_a)$ is 1, and $|E(\mathbb{C}(t))_{\mathrm{tor}}| = 2$, the above formula becomes

$$1 = \frac{\mathrm{disc}\,T(E_a) \cdot h(R)}{4},$$

where R is a generator for the infinite section.

By Definition 7.3 from [Shi90],

$$(7) \qquad \mathrm{disc}\,T(E_a) = \prod_v m_v^{(1)},$$

where $m_v^{(1)}$ is the number of simple components of the corresponding singular fiber. We have $m_v^{(1)} = n$ if the type is I_n and $m_v^{(1)} = 4$ if the type is I_0^*. We thus get

$$\mathrm{disc}\,T(E_a) = 16.$$

Therefore we conclude that

$$h(R) = \frac{1}{4}.$$

Our goal is to prove that P_a is a generator of $E_a(\mathbb{C}(t))$. We will do this by showing that its height is exactly $\frac{1}{4}$.

By the explicit formula for the height ([SS10], Section 11.8), for any section $P \in E_a(\mathbb{C}(t))$ we have

$$(8) \qquad h(P) = 2\chi(E_a) + 2\overline{P} \cdot \overline{O} - \sum_v \mathrm{contr}_v(P).$$

We remark that P_a never intersects the zero section $O = [0 : 1 : 0]$ (for t finite, this is clear, and we will soon see that P_a becomes the point $(8s, 8s(1 + a))$ upon the change $s = \frac{1}{t}$ and therefore there is no intersection at infinity either). Therefore $\overline{P_a} \cdot \overline{O} = 0$.

By looking at Table 4 in Section 11.9 of [SS10], we get that $\mathrm{contr}_v(P_a) = 1$ for I_0^* (corresponding to $t = 0$), and $\mathrm{contr}_v(P_a) = 0$ for I_1 (corresponding to $t = \sigma_\pm$). It remains to compute the correction term for $t = \infty$. In order to do this, consider the change $s = \frac{1}{t}$, $Y' = s^3 Y$, $X' = s^2 X$, we have

$$(9) \qquad Y'^2 = X'^3 + (4(a - 3)s + (1 + a)^2)X'^2 + 32(1 - a)s^2 X',$$

and the section P_a becomes

$$P_a = (8s, 8(1 + a)s).$$

We need the following result.

Theorem 8 (Néron [N64]). *Let E_s be an elliptic curve defined over $\mathbb{C}[s]$ given by a Weierstrass model, and denote by v the s-adic valuation. Suppose that E_0 has a double point with distinct tangents and $v(j(E_s)) = -m < 0$ (this happens if and only if E_0 is singular of type I_m). Then, for every integer $l > m/2$, there exists a Weierstrass model $\mathcal{E}_s$ deduced from E_s by a transformation of the form*

$$\begin{aligned} X &= x + qz, \\ Y &= y + ux + rz, \\ Z &= z, \end{aligned}$$

with $q, r, u \in \mathbb{C}[s]$. The Weierstrass model $\mathcal{E}_s$ is given by

$$(10) \qquad Y^2 Z + \lambda XYZ + \mu YZ^2 = X^3 + \alpha X^2 Z + \beta XZ^2 + \gamma Z^3,$$

with coefficients satisfying

$$(11) \qquad v(\lambda^2 + 4\alpha) = 0, \quad v(\mu) \geq l, \quad v(\beta) \geq l, \quad v(\gamma) = m, \quad \text{and} \quad v(j(\mathcal{E}_s)) = -m.$$

We follow the exposition of [Ber10, BFF+13]. A singular fiber of type I_m over $s = 0$ is composed by nonsingular rational curves $\Theta_{0,0}, \Theta_{0,1}, \ldots, \Theta_{0,m-1}$. When $m = 2h$, the configuration of these curves can be found in $(\mathbb{P}^2)^h$ with a point $[X : Y : Z] \in E_a$ over $s = 0$ corresponding to the point

$$[X : Y : Z^{(1)}] \times [X : Y : Z^{(2)}] \times \cdots \times [X : Y : Z^{(h)}] \in (\mathbb{P}^2)^h,$$

where $[X : Y : Z^{(i+1)}] = [X : Y : sZ^{(i)}]$.

If $[X : Y : Z]$ satisfies equation (10), then $[X : Y : Z^{(1)}]$ satisfies equation

$$Y^2 Z^{(1)} + \lambda XYZ^{(1)} + (\mu/s)Y(Z^{(1)})^2 = sX^3 + \alpha X^2 Z^{(1)} + (\beta/s)X(Z^{(1)})^2 + (\gamma/s^2)(Z^{(1)})^3.$$

Under conditions (11) together with $m = 2h \geq 4$, the equation above simplifies upon evaluation at $s = 0$ to

$$Z^{(1)}(Y^2 + \lambda_0 XY - \alpha_0 X^2) = 0,$$

where the subscript 0 indicates evaluation at $s = 0$.

We remark that in our case $\lambda = 0$ and therefore the equation above becomes

$$Z^{(1)}(Y - \nu X)(Y + \nu X) = 0,$$

where $\alpha_0 = \nu^2$.

The components that will be relevant for us are given by

$$\Theta_{0,0} = [X : Y : 0] \times \cdots \times [X : Y : 0],$$
$$\Theta_{0,1} = [X : \nu X : Z] \times [1 : \nu : 0] \times \cdots \times [1 : \nu : 0].$$

Back to equation (9), we make the change of variables $X' = \frac{X_1}{(a+1)^2} + \frac{16(a-1)s^2}{(a+1)^2}$, $Y' = \frac{Y_1}{(a+1)^3}$ and rewrite in projective coordinates:

$$Y_1^2 Z = X_1^3 + (48(a-1)s^2 + 4(a-3)(a+1)^2 s + (a+1)^4)X_1^2 Z$$
$$+ 128(a-1)s^3(6(a-1)s + (a-3)(a+1)^2)X_1 Z^2$$
$$+ 256(a-1)^2 s^4(16(a-1)s^2 + 4(a-3)(a+1)^2 s - (a+1)^4)Z^3.$$

It is straightforward to check that the above Weierstrass form satisfies the conditions of Theorem 8.

Thus, making the change $Z^{(1)} = sZ$, we obtain that $[X_1 : Y_1 : Z^{(1)}]$ satisfies

(12) $$Z^{(1)}(Y_1 - (a+1)^2 X_1)(Y_1 + (a+1)^2 X_1) = 0 \text{ at } s = 0.$$

The same change of variables applied to the section P_a gives in $[X_1 : Y_1 : Z]$ coordinates,

$$\left[8(1+a)^2 s - 16(a-1)s^2 : 8(1+a)^4 s : 1\right],$$

and in $[X_1 : Y_1 : Z^{(1)}]$ coordinates,

$$\left[8(1+a)^2 - 16(a-1)s : 8(1+a)^4 : 1\right],$$

This corresponds to $[8(1+a)^2 : 8(1+a)^4 : 1]$ in the conic (12) and indicates intersection with $\Theta_{0,1}$.

Therefore, according to the Table 4 in Section 11.9 of [SS10], $\text{contr}_\infty(P_a) = \frac{1 \cdot 3}{4} = \frac{3}{4}$. Back to equation (8), we have

$$h(P_a) = 2 \cdot 1 + 2 \cdot 0 - 1 - 2 \cdot 0 - \frac{3}{4} = \frac{1}{4}.$$

Therefore P_a is a generator for the Mordell–Weil group of the surface.

$\square$

Since $E_a(\mathbb{Q}(t)) \subset E_a(\mathbb{C}(t))$, and $P_a(t)$, $(0,0)$ are also defined over $\mathbb{Q}(t)$, we deduce the same conclusions of Theorem 7 for $E_a(\mathbb{Q}(t))$.

Silverman's Specialization Theorem ([Sil94], Theorem 11.4) tells us that for all but finitely many values of t the rank of $E_{a,t}(\mathbb{Q})$ is greater than or equal to that of $E_a(\mathbb{Q}(t))$. We then conclude that

$$\text{rk} E_{a,t}(\mathbb{Q}) \geq 1$$

for all but finitely many values of t.

We finish this section by mentioning that a similar analysis can be made by fixing the value of t and thinking of E_t as an rational elliptic surface over $\mathbb{C}(a)$. It can be proven, for instance, that the Mordell–Weil

rank is again 1. This approach is less natural from the context of our geometric problem, therefore we do not include the discussion here.

5. Proof of Theorem 2

In this section we impose the additional restriction that $\cos \omega \in \mathbb{Q}$. Following an idea similar to what is done in the right-triangle case, where both $\sin \omega$ and $\cos \omega$ are parametrized in terms of $\tan(\omega/2)$ and $\tan(\omega/2)$ is required to be rational, consider a triangle ABC with angles θ and ω at A and B respectively, take the bisector of ω which crosses the side AC at B' and consider the quotient $\tau = \frac{|AB'|}{|AB|}$ (see Figure 3). By the law of sines,

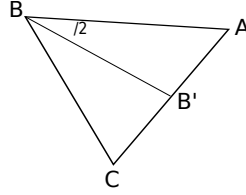$$\tau = \frac{\sin(\omega/2)}{\sin(\omega/2 + \theta)}.$$

FIGURE 3. Construction of $\tau = \frac{\sin(\omega/2)}{\sin(\omega/2+\theta)}$.

By the law of cosines,

$$|BB'|^2 = |AB'|^2 + |AB|^2 - 2a|AB'||AB| \Rightarrow \frac{|BB'|^2}{|AB|^2} = \tau^2 + 1 - 2a\tau,$$

and

$$|AB'|^2 = |BB'|^2 + |AB|^2 - 2\cos(\omega/2)|BB'||AB| \Rightarrow \tau^2 = \frac{|BB'|^2}{|AB|^2} + 1 - 2\cos(\omega/2)\frac{|BB'|}{|AB|}.$$

Combining the above equations, we get,

$$\cos(\omega/2) = \frac{1 - a\tau}{\sqrt{\tau^2 - 2a\tau + 1}}.$$

Therefore,

$$\sin \omega = \frac{2\sqrt{1 - a^2}\tau(1 - a\tau)}{\tau^2 - 2a\tau + 1}, \qquad \cos \omega = \frac{(2a^2 - 1)\tau^2 - 2a\tau + 1}{\tau^2 - 2a\tau + 1}.$$

Notice that we have $|a| < 1$ as before. The fact that $\frac{\omega}{2} < \frac{\pi}{2}$ implies that $a\tau < 1$ and $2a\tau < \tau^2 + 1$ (so that $\cos(\omega/2)$ is a positive real number). Since $\sin \omega$ is positive, $\tau > 0$.

Before we continue, we remark that we could have parametrized the rational solutions to the equation

$$(1 - a^2)x^2 + y^2 = z^2.$$

This leads to

$$\sin \omega = \frac{2\sqrt{1 - a^2}\lambda}{\lambda^2 + 1 - a^2}, \qquad \cos \omega = \frac{\lambda^2 - 1 + a^2}{\lambda^2 + 1 - a^2}.$$

Notice that the change of variables

$$\lambda = \frac{1 - a\tau}{\tau}$$

allows us to go between both formulations. Indeed, this implies that

$$\lambda = \tan(\omega/2)\sin \theta.$$

We will continue with the formulation with τ because it is more meaningful from the geometric point of view. [1]

---

[1] Another reason to work with τ is that the coefficients in the elliptic curve $F_{a,\tau}$ defined by (13) are polynomials of even degree in τ as opposed to the elliptic curve that we would obtain working with λ. Having degree even polynomials as coefficients simplifies some of the arguments used in the 2-descent from Section 7.

Proposition 9. *Let $a, \tau \in \mathbb{Q}$, $|a| < 1$, such that $0 < \tau$, $a\tau < 1$, and $2a\tau < \tau^2 + 1$. If the elliptic curve*

$$F_{a,\tau} : Y^2 = X^3 + \tau(1 - a\tau)(2(a-3)(\tau^2 - 2a\tau + 1) + (1+a)^2\tau(1 - a\tau))X^2$$

(13)
$$+ 8(1-a)\tau^2(1 - a\tau)^2(\tau^2 - 2a\tau + 1)^2 X$$

has infinitely many points on $F_{a,\tau}(\mathbb{Q})$ satisfying the conditions

(14)
$$2(1-a)\tau(1 - a\tau)(\tau^2 - 2a\tau + 1) < X < 4\tau(1 - a\tau)(\tau^2 - 2a\tau + 1),$$

(15)
$$|Y| < (1+a)\tau(1 - a\tau)X,$$

then there exist infinitely many pairs of integral θ-triangle and integral ω-parallelogram such that $\cos\theta = a$ and $\sin(\omega/2)/\sin(\omega/2 + \theta) = \tau$ with common area and common perimeter.

Proof. We have that $\frac{\sin\omega}{\sin\theta} = \frac{2\tau(1-a\tau)}{\tau^2 - 2a\tau + 1}$. With this change of variables, system (1) becomes

$$\begin{cases} (n^2 - m^2)m(n - am) = \frac{2\tau(1-a\tau)}{\tau^2 - 2a\tau + 1}pq, \\ \\ (n - am)(n + m) = p + q. \end{cases}$$

Combining the equations above,

$$(n - am)^2(n + m)^2 - 4\frac{\tau^2 - 2a\tau + 1}{2\tau(1 - a\tau)}(n^2 - m^2)m(n - am) = (p + q)^2 - 4pq = (p - q)^2.$$

By considering the change of variables

$$\begin{cases} Y = 2(1 - a^2)\tau^2(1 - a\tau)^2(\tau^2 - 2a\tau + 1)\frac{p-q}{(n-am)^2}, \\ \\ X = 2(1 - a)\tau(1 - a\tau)(\tau^2 - 2a\tau + 1)\frac{n+m}{n-am}, \end{cases}$$

we obtain the Weierstrass form (13).

Now given a point $(X, Y) \in F_{a,\tau}(\mathbb{Q})$, we can take

$$\begin{cases} x = X(1 - a^2)(4\tau(1 - a\tau)(\tau^2 - 2a\tau + 1) - X), \\ \\ y = 4(1 - a^2)\tau(1 - a\tau)(\tau^2 - 2a\tau + 1)(X - 2(1 - a)\tau(1 - a\tau)(\tau^2 - 2a\tau + 1)), \end{cases}$$

and

$$p, q = (1 - a^2)(\tau^2 - 2a\tau + 1)((a + 1)\tau(1 - a\tau)X \pm Y).$$

Recall that we have the conditions $|a| < 1$, $0 < \tau$, $a\tau < 1$ and $2a\tau < \tau^2 + 1$. In addition, for the solution to have geometric meaning we need $x, y, p, q > 0$. This happens if

$$|Y| < (1 + a)\tau(1 - a\tau)X, \qquad 0 < X(4\tau(1 - a\tau)(\tau^2 - 2a\tau + 1) - X),$$

and

$$2(1 - a)\tau(1 - a\tau)\left(\tau^2 - 2a\tau + 1\right) < X.$$

These restrictions can be condensed as conditions (14) and (15).

$\square$

Lemma 10. *For a given $a \in \mathbb{Q}$ with $|a| \neq 1$,*

$$Q_{a,\tau} = (4\tau(1 - a\tau)(\tau^2 - 2a\tau + 1), 4(1 + a)\tau^2(1 - a\tau)^2(\tau^2 - 2a\tau + 1))$$

is a point of infinite order in $F_{a,\tau}$ for almost every τ. In particular, $F_{a,\tau}$ has positive rank for all but finitely many $\tau \in \mathbb{Q}$.

Proof. The discriminant of $F_{a,\tau}$ is given by

$$\Delta_{a,\tau} = 2^{10}(a - 1)^2(a + 1)^2\tau^6(a\tau - 1)^6(\tau^2 - 2a\tau + 1)^4$$
$$\times \left[(a^4 + 2a^3 - 3a^2 + 12a + 4)\tau^4 + 2(3a^3 - 14a^2 - 7a - 6)\tau^3 + (5a^2 + 38a + 9)\tau^2 - 12(a + 1)\tau + 4\right].$$

We have an elliptic curve as long as $a \neq \pm 1$ and τ is not a root of the discriminant.

The point $Q_{a,\tau}$ can be found by numerical experimentation. We also have the point $(0,0)$ of order 2. It can be proven that $Q_{a,\tau}$ has infinite order by computing $kQ_{a,\tau}$ and $kQ_{a,\tau} + (0,0)$ for $k = 1,\dots,4$ as it was done in Lemma 5.

$\square$

Proof of Theorem 2. This proof uses Theorem 6 and follows the same lines as the proof of Theorem 1. $\square$

One can consider particular cases for $F_{a,\tau}$ in the same way as we considered particular cases of $E_{a,t}$. However, they are all included in the cases discussed in Section 3, so we will not provide the details here. One can also search for examples with all the possible torsion groups as done in Subsection 4.1. The methods are the same, so we will not repeat them here.

6. The rank of $F_a(\mathbb{C}(\tau))$

The treatments of $F_a(\mathbb{C}(\tau))$ and $F_a(\mathbb{Q}(\tau))$ are different from that of $E_a(\mathbb{C}(t))$. While E_a is a rational elliptic surface, F_a is a K3-surface, and the bound for the rank of the Néron–Severi lattice ρ is less optimal. The motivation for considering the elliptic surfaces is, as in Section 4.2, the application of Silverman's Specialization Theorem that gives a lower bound for the rank of $F_{a,\tau}(\mathbb{Q})$ for all but finitely many values of τ provided that we know the rank of $F_a(\mathbb{Q}(\tau))$. In this section, we will consider the structure over $\mathbb{C}(\tau)$. We will discuss the structure over $\mathbb{Q}(\tau)$ in the next section.

We consider

$$F_a : Y^2 = X^3 + \tau(1-a\tau)(2(a-3)(\tau^2 - 2a\tau + 1) + (1+a)^2\tau(1-a\tau))X^2$$
$$+ 8(1-a)\tau^2(1-a\tau)^2(\tau^2 - 2a\tau + 1)^2 X,$$

with discriminant

$$\Delta_a = 2^{10}(a-1)^2(a+1)^2\tau^6(a\tau-1)^6(\tau^2 - 2a\tau + 1)^4$$
$$\times \left[(a^4 + 2a^3 - 3a^2 + 12a + 4)\tau^4 + 2(3a^3 - 14a^2 - 7a - 6)\tau^3 + (5a^2 + 38a + 9)\tau^2 - 12(a+1)\tau + 4\right].$$

and a point

$$Q_a = (4\tau(1-a\tau)(\tau^2 - 2a\tau + 1), 4(1+a)\tau^2(1-a\tau)^2(\tau^2 - 2a\tau + 1)),$$

which is generically of infinite order.

Recall that $|a| < 1$ so that $\Delta_a \neq 0$ except for finitely many values of τ. The coefficients a_2 and a_4 of F_a are polynomials in τ of degrees 4 and 8 respectively (3 and 6 for $a = 0$). By Section 4.10 in [SS10] (or 5.13 in [SS]), we have an elliptic K3-surface.

Moreover, the Euler characteristic is $\chi(F_a) = 2$ by the discussion in Sections 5.12 and 5.13 of [SS], and the rank of the Néron–Severi lattice satisfies $\rho(F_a) \leq 20$ ([SS10], 13.1).

The singularities of F_a are at $\tau = 0$, $\tau = \infty$, $\tau = \frac{1}{a}$, the roots of $\tau^2 - 2a\tau + 1$ and the roots of the remaining polynomial factor of Δ_a that has degree 4 in both τ and a. Assume first that $a \neq 0$. For $\tau = 0$ and $\tau = \frac{1}{a}$, we get singularities of type I_0^*. For $\tau = \infty$, the type is I_0 and is therefore nonsingular. For the roots of $\tau^2 - 2a\tau + 1$ the type is I_4. For the other roots the type is I_1. By Shioda–Tate formula,

$$\rho(F_a) = \operatorname{rk} F_a(\mathbb{C}(\tau)) + 2 + 2(5-1) + 2(4-1) + 4 \cdot (1-1) = \operatorname{rk} F_a(\mathbb{C}(\tau)) + 16,$$

and the rank $\operatorname{rk} F_a(\mathbb{C}(\tau))$ is bounded by 4.

When $a = 0$ the singularity at $\tau = \infty$ is of type I_0^* and the same bound applies.

Later in Section 7 we will see that the rank of $F_{\frac{1}{2}}(\mathbb{Q}(\tau))$ is 1 and that the rank of the elliptic threefold $F(\mathbb{Q}(a,\tau))$ is 1.

Formula (6) implies

$$(16) \qquad |\operatorname{disc} \operatorname{NS}(F_a)| = \frac{|\operatorname{disc} T(F_a) \cdot \operatorname{disc} \operatorname{MWL}(F_a)|}{|F_a(\mathbb{C}(t))_{\operatorname{tor}}|^2}.$$

By a similar argument to the one for E_a, one can see that there are no points of order 2 over $\mathbb{C}(t)$ other than $(0,0)$ and one can also see that $(0,0)$ can not be written as $2P$ or $3P$ over $\mathbb{C}(t)$. By Table (4.5) in p. 264 of [MP89], we get that $F_a(\mathbb{C}(t))_{\operatorname{tor}} \cong \mathbb{Z}/2\mathbb{Z}$.

Equation (7) implies

$$\operatorname{disc} T(F_a) = 64.$$

Therefore equation (16) becomes

$$|\text{disc}\,\text{NS}(F_a)| = 16|\text{disc}\,\text{MWL}(F_a)|.$$

Notice that we have a morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1$$

$$[\tau : 1] \to [2\tau(1 - a\tau) : \tau^2 - 2a\tau + 1]$$

yielding a base change

$$F_a \to E_a$$

$$[X : Y : Z] \to \left[\left(\frac{2}{\tau^2 - 2a\tau + 1} \right)^2 X : \left(\frac{2}{\tau^2 - 2a\tau + 1} \right)^3 Y : Z \right],$$

which in particular sends $Q_a(\tau)$ to $P_a(t)$.

By Proposition 11.14 of [SS10], since $\deg \varphi = 2$,

$$h(Q_a) = 2h(P_a) = \frac{1}{2}.$$

If we assume that $\text{rk}F_a(\mathbb{C}(\tau)) = 1$, *then* $|\text{disc}\,\text{MWL}(F_a)|$ *is given by* $h(R)$, *where* R *is a generator, and* $\frac{1}{2} = h(Q_a) = n^2 h(R)$. However, since $|\text{disc}\,\text{NS}(F_a)|$ is an integer, then $16h(R) \in \mathbb{Z}$, and $\frac{8}{n^2} \in \mathbb{Z}$. This means that $n = 1$ or $n = 2$. One can then show that that it is not possible to write $Q_a(\tau) = 2P$ in $\mathbb{C}(\tau)$ and from that one can conclude that $Q_a(\tau)$ is a generator for $F_a(\mathbb{C}(\tau))$.

We close this section by mentioning that another possible direction of study would be to directly consider the elliptic threefold $F(\mathbb{C}(a, \tau))$. Indeed, in this case, a Shioda–Tate formula was given by Wazir ([Waz04], Corollary 3.2).

7. The ranks of $F(\mathbb{Q}(\tau, a))$ and of $F_a(\mathbb{Q}(\tau))$

Because our study of the structure of $F_a(\mathbb{C}(\tau))$ was not entirely conclusive since we could not completely determine $\text{rk}F_a(\mathbb{C}(\tau))$, we will directly discuss $F_a(\mathbb{Q}(\tau))$ here.

We will use the method of the 2-descent to determine the 2-Selmer group over $\mathbb{Q}(\tau)$ in a similar construction to the one described by Bremner in Section 3.2 of [Bre81], which is based on Lemma 8 of [BSD65].

In general, let $F : y^2 = x(x^2 + ax + b)$ with $b, a^2 - 4b \neq 0$ be an elliptic curve over a field K ($\text{char}(K) \neq 2$) with a K-rational point of order 2. Let $\hat{F} : \hat{y}^2 = \hat{x}(\hat{x}^2 - 2a\hat{x} + (a^2 - 4b))$, also an elliptic curve. We have the degree 2 isogenies ϕ and $\hat{\phi}$

$$F \xrightarrow{\phi} \hat{F} \xrightarrow{\hat{\phi}} F,$$

given by $\phi, \hat{\phi} : (0, 0), O \to O$ and for $x, \hat{x} \neq 0$,

$$\phi(x, y) = \left(\frac{y^2}{x^2}, \frac{y(b - x^2)}{x^2} \right), \quad \hat{\phi}(\hat{x}, \hat{y}) = \left(\frac{\hat{y}^2}{4\hat{x}^2}, \frac{\hat{y}((a^2 - 4b) - \hat{x}^2)}{8\hat{x}^2} \right)$$

and such that $\phi \circ \hat{\phi}$ corresponds to multiplication by 2 in F. Then one can use $\hat{F}(K)/\phi(F)(K)$ and $F(K)/\hat{\phi}(\hat{F})(K)$ to find generators for $F(K)/2F(K)$. The points of $\hat{F}(K)/\phi(F)(K)$ correspond to classes $\delta \in K^*/(K^*)^2$ such that the equation

$$(17) \qquad \delta T^2 = \delta^2 R^4 - 2a\delta R^2 S^2 + (a^2 - 4b)S^4,$$

has a nontrivial solution (T, R, S) (and this corresponds to the point $\hat{x} = \delta \frac{R^2}{S^2}, \hat{y} = \delta \frac{RT}{S^3}$ in $\hat{F}$) and those of $F(K)/\hat{\phi}(F)(K)$ correspond to classes $\delta \in K^*/(K^*)^2$ giving nontrivial solutions to

$$(18) \qquad \delta T^2 = \delta^2 R^4 + a\delta R^2 S^2 + bS^4.$$

Of course in general the quartics (17), (18) are not expected to satisfy the local-global principle. If they have local solutions everywhere without having global solutions they give rise to points in the 2-Selmer group rather than $F(K)/2F(K)$.

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In our case, we have to investigate the following equations

(19) $\delta T^2 = \delta^2 R^4 - 2\delta\tau(1 - a\tau)(2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))R^2S^2$
$$+ (a + 1)^2\tau^2(1 - a\tau)^2\big[(a^4 + 2a^3 - 3a^2 + 12a + 4)\tau^4 + 2(3a^3 - 14a^2 - 7a - 6)\tau^3$$
$$+ (5a^2 + 38a + 9)\tau^2 - 12(a + 1)\tau + 4\big]S^4,$$

(20) $\delta T^2 = \delta^2 R^4 + \delta\tau(1 - a\tau)(2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))R^2S^2$
$$+ 8(1 - a)\tau^2(1 - a\tau)^2(\tau^2 - 2a\tau + 1)^2S^4,$$

where $\delta, R, S, T \in \mathbb{Q}(\tau)$ (or $\mathbb{Q}(a, \tau)$ if we keep a as a variable). Without loss of generality, we can assume that $\delta, R, S, T \in \mathbb{Z}[\tau]$ (or $\mathbb{Z}[a, \tau]$), δ is square-free, and that $(\delta R, S) = 1$. It is immediate to see that in this case δ divides the coefficient of S^4.

We start by considering first the case of F as the elliptic threefold, namely, we think of F as an elliptic curve over $\mathbb{Q}(a, \tau)$.

Theorem 11. *The rank of $F(\mathbb{Q}(a, \tau))$ is 1 and its Mordell–Weil group is generated by the point $Q(a, \tau)$.*

Proof. Consider equation (19). Then $\delta \mid (a+1)\tau(1-a\tau)\Phi$, where $\Phi = (a^4 + 2a^3 - 3a^2 + 12a + 4)\tau^4 + 2(3a^3 - 14a^2 - 7a - 6)\tau^3 + (5a^2 + 38a + 9)\tau^2 - 12(a + 1)\tau + 4$.

By reducing modulo τ, we obtain that
$$\delta T^2 \equiv \delta^2 R^4 \bmod (\tau).$$

Therefore, $\delta \equiv \square \bmod (\tau)$ (δ is a square modulo τ), or τ divides both R and T. Writing $T = T_1\tau$, dividing the equation by τ^2, and reducing modulo τ again, we have
$$\delta T_1^2 \equiv 4S^4 \bmod (\tau).$$

This implies again that $\delta \equiv \square \bmod (\tau)$ or $\tau \mid S$, but this last condition is a contradiction since $(\delta R, S) = 1$. Thus, we conclude that $\delta \equiv \square \bmod (\tau)$.

Now equation (19) can also be written as
$$\delta T^2 = (\delta R^2 - \tau(1 - a\tau)(2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))S^2)^2$$
$$- 32(1 - a)\tau^2(1 - a\tau)^2(\tau^2 - 2a\tau + 1)^2S^4.$$

Assume that $\deg_\tau(\delta)$ is odd. Then the right-hand side has to have odd degree and this can only happen if there is some cancellation. The term involving R inside the parenthesis has odd degree, while the term involving S has even degree. Therefore, there is no cancellation inside the parenthesis. By comparing the degree of the other term, the only way to have cancellation is that the main coefficients with S cancel each other. The main coefficient coming from the parenthesis is $a^2(a+3)^2(a^2-a+2)^2\tau^8$ while the main coefficient from the second term is $32(a-1)a^2\tau^8$, and they do not cancel (another way of seeing this property is that the main coefficient of S^4 in equation (19) has degree 8 in τ, so there is no cancellation involved). Therefore, we get a contradiction and $\deg_\tau(\delta)$ must be even.

Assume that $\tau \mid \delta$. Then the degree considerations imply that $\tau(1 - a\tau) \mid \delta$. From this we deduce that $\tau(1 - a\tau) \mid T$. Write $\delta = \delta_1\tau(1 - a\tau)$, $T = T_1\tau(1 - a\tau)$. Dividing by $\tau^2(1 - a\tau)^2$ gives
$$\delta_1\tau(1 - a\tau)T_1^2 = (\delta_1 R^2 - (2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))S^2)^2$$
$$- 32(1 - a)(\tau^2 - 2a\tau + 1)^2S^4.$$

Reducing modulo τ we get that $2(1 - a) \equiv \square \bmod (\tau)$ or τ divides S, R, both contradictions.

Therefore, $\delta = 1$ or $\delta = \Phi$. These lead to O and $(0, 0)$ in $\hat{F}/\phi(F)$, which in turn lead to O in $F/2F$.

Now consider equation (20). Then $\delta \mid 2(1 - a)\tau(1 - a\tau)(\tau^2 + 1 - 2a\tau)$. By reducing modulo τ, we obtain that $\delta \equiv \square \bmod (\tau)$ or $2(1 - a)\delta \equiv \square \bmod (\tau)$ as before. The equation can also be written as
$$4\delta T^2 = \big(2\delta R^2 + \tau(1 - a\tau)(2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))S^2\big)^2 - (a + 1)^2\tau^2(1 - a\tau)^2\Phi S^4.$$

As before, we can conclude that $\deg_\tau(\delta)$ is even.

If $\tau \mid \delta$, then $\tau(1 - a\tau)$ divides both δ and T. As before, we have,
$$4\delta_1\tau(1 - a\tau)T_1^2 = \big(2\delta_1 R^2 + (2(a - 3)(\tau^2 - 2a\tau + 1) + (1 + a)^2\tau(1 - a\tau))S^2\big)^2 - (a + 1)^2\Phi S^4.$$

On the one hand, reduce modulo τ. We obtain

$$(\delta_1 R^2 - 4S^2)(\delta_1 R^2 + 2(a-1)S^2) \equiv 0 \bmod (\tau).$$

Since we can not have that τ divides both R and S, we conclude that either $\delta_1 \equiv \square \bmod (\tau)$ or $2(1-a)\delta_1 \equiv \square \bmod (\tau)$.

On the other hand, reduce modulo $1 - a\tau$. We obtain

$$\left(\delta_1 R^2 + 4(1-\tau^2)S^2\right)\left(\delta_1 R^2 + 2(1-a)(1-\tau^2)S^2\right) \equiv 0 \bmod (1 - a\tau).$$

Since we can not have that $1-a\tau$ divides both R and S, we conclude that either $(\tau^2-1)\delta_1 \equiv \square \bmod (1-a\tau)$ or $2(1-a)(\tau^2-1)\delta_1 \equiv \square \bmod (1-a\tau)$.

Therefore, the possible values for δ are $1, 2(1-a), \tau^2 - 2a\tau + 1, 2(1-a)(\tau^2 - 2a\tau + 1), \tau(1-a\tau)(\tau^2 - 2a\tau + 1), 2(1-a)\tau(1-a\tau)(\tau^2 - 2a\tau + 1)$.

Combining both equations we obtain at most 7 elements in $F/2F$, and F has a point of order 2 which is not the double of point. This implies that the rank is at most 1. We remark that the point $Q(a,\tau)$ comes from the solution $\delta = \tau(1-a\tau)(\tau^2 - 2a\tau + 1)$, with $R = 2, S = 1, T = 2(1+a)\tau(1-a\tau)$. Therefore, it is a generator.

$\square$

Theorem 12. *The rank of $F_{\frac{1}{2}}(\mathbb{Q}(\tau))$ is 1 and its Mordell–Weil group is generated by the point $Q_{\frac{1}{2}}(\tau)$.*

Proof. Setting $\tau = 2\sigma$, equation (19) becomes

$$\begin{aligned}
\delta T^2 &= \delta^2 R^4 - 2\delta\sigma(\sigma-1)(49\sigma^2 - 29\sigma + 10)R^2 S^2 + 9\sigma^2(\sigma-1)^2(9\sigma^2 - 5\sigma + 2)(17\sigma^2 - 13\sigma + 2)S^4 \\
&= (\delta R^2 - \sigma(\sigma-1)(49\sigma^2 - 29\sigma + 10)S^2)^2 - 64\sigma^2(\sigma-1)^2(4\sigma^2 - 2\sigma + 1)^2 S^4.
\end{aligned}$$

We have that $\delta \mid 3\sigma(\sigma-1)(9\sigma^2 - 5\sigma + 2)(17\sigma^2 - 13\sigma + 2)$.

By reducing modulo σ as before, we have that $\delta \equiv \square \bmod (\sigma)$. As in the previous case, we can also deduce that $\deg(\delta)$ must be even.

If $\sigma \mid \delta$, then $\sigma(\sigma-1) \mid \delta, T$. As before, this leads to

$$\delta_1(\sigma-1)\sigma T_1^2 = (\delta_1 R^2 - (49\sigma^2 - 29\sigma + 10)S^2)^2 - 64(4\sigma^2 - 2\sigma + 1)^2 S^4.$$

Reducing modulo σ, we get,

$$(\delta_1 R^2 - 2S^2)(\delta_1 R^2 - 18S^2) \equiv 0 \bmod (\sigma).$$

Then we must have that $2\delta_1 \equiv \square \bmod (\sigma)$.

Thus, our options are $\delta = 1, \sigma(\sigma-1)(9\sigma^2 - 5\sigma + 2), \sigma(\sigma-1)(17\sigma^2 - 13\sigma + 2), (9\sigma^2 - 5\sigma + 2)(17\sigma^2 - 13\sigma + 2)$. This gives at most one point of infinite order. Indeed, when $\delta = \sigma(\sigma-1)(9\sigma^2 - 5\sigma + 2)$ we obtain the nontrivial solution $R = S = 1, T = 8\sigma(\sigma-1)$ which corresponds to the point $2Q_{\frac{1}{2}} + (0,0)$, equivalent to $(0,0)$ in $F/2F$. The other values of δ yield O, $(0,0)$ and $2Q_{\frac{1}{2}}$. Therefore we do not obtain points of infinite order in this case.

Setting $\tau = 2\sigma$, equation (20) becomes

$$\begin{aligned}
\delta T^2 &= \delta^2 R^4 + \delta\sigma(\sigma-1)(49\sigma^2 - 29\sigma + 10)R^2 S^2 + 16\sigma^2(\sigma-1)^2(4\sigma^2 - 2\sigma + 1)^2 S^4 \\
4\delta T^2 &= (2\delta R^2 + \sigma(\sigma-1)(49\sigma^2 - 29\sigma + 10)S^2)^2 - 9\sigma^2(\sigma-1)^2(9\sigma^2 - 5\sigma + 2)(17\sigma^2 - 13\sigma + 2)S^4.
\end{aligned}$$

We have $\delta \mid 2\sigma(\sigma-1)(4\sigma^2 - 2\sigma + 1)$.

By reducing modulo σ as before, we have that $\delta \equiv \square \bmod (\sigma)$. A similar argument reducing modulo $\sigma - 1$ leads to $\delta \equiv \square \bmod (\sigma - 1)$. As in the previous case, we obtain a contradiction if $\deg(\delta)$ is odd.

If $\sigma \mid \delta$, then $\sigma(\sigma-1) \mid \delta, T$. As before,

$$4\delta_1\sigma(\sigma-1)T_1^2 = (2\delta_1 R^2 + (49\sigma^2 - 29\sigma + 10)S^2)^2 - 9(9\sigma^2 - 5\sigma + 2)(17\sigma^2 - 13\sigma + 2)S^4.$$

On the one hand, reducing modulo σ,

$$(\delta_1 R^2 + 2S^2)(\delta_1 R^2 + 8S^2) \equiv 0 \bmod (\sigma)$$

and we must have $-2\delta_1 \equiv \square \bmod (\sigma)$.

On the other hand, evaluating at $\sigma = 1$,

$$(\delta_1 R^2 + 6S^2)(\delta_1 R^2 + 24S^2) \equiv 0 \bmod (\sigma - 1)$$

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and we must have $-6\delta_1 \equiv \square \bmod (\sigma - 1)$.

This implies that $\delta = 1, -2\sigma(\sigma - 1)(4\sigma^2 - 2\sigma + 1)$. Indeed, $\delta = 1$ produces $(0, 0)$, while $\delta = -2\sigma(\sigma - 1)(4\sigma^2 - 2\sigma + 1)$ gives the solution $R = S = 1, T = 3\sigma(\sigma - 1)$ yielding $-Q_{\frac{1}{2}} + (0, 0)$. Thus, we have that $Q_{\frac{1}{2}}$ is a generator. $\qquad\square$

We have tried to prove version of Theorem 12 for $a = 0$ and $a = -\frac{1}{2}$. However, in both cases we are only able to bound the rank by 2.

As a final note, this problem could be potentially approached by Nagao's conjecture [Nag97], which predicts that the rank of F_a over $\mathbb{Q}(\tau)$ can be obtained as a certain limit involving a weighted average of the p-coefficients of the L-function associated to the fibers. Rosen and Silverman [RS98] proved this conjecture for rational elliptic surfaces and in general when Tate's conjecture holds. This includes the case of certain $K3$-surfaces. In fact, the analytic version of Nagao's conjecture is true for all $K3$-surfaces defined over $\mathbb{Q}$ [HP05]. Numerical experimentation of Nagao's conjecture seems to support that $\mathrm{rk}F_a(\mathbb{Q}(\tau)) = 1$ for $a = 0, -\frac{1}{2}$.

Acknowledgments

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