Richardson varieties and Khovanov Rozansky homology

following Galashin and Lam, arXiv:2012.09745(v3)

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I would like to give a series of talks that outlines the main statements of [GL20] and then meanders through the various objects which appear.

My plan is to give one talk on [GL20], one or two talks on cluster theory (cluster algebras and examples, then cluster varieties and their cohomology), and one talk on Hogancamp's and Muller+Speyer's recurrences.

[⋆I will not give the recurrences talk, as I talked a lot about cluster stuff already.⋆]

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1 Galashin+Lam

1.1 Main theorem and applications

The main theorem of [GL20] expresses KhR homology of certain links as honest cohomology of varieties:

Theorem 1.1.1 ([GL20, Thm 1.16, 1.17]). *Let $v \leq w \in S_n$ and let $\beta_{v,w} = \beta(w)\beta(v)^{-1}$ be a Richardson braid. Let $\beta = \widehat{\beta_{v,w}}$ be the corresponding Richardson link and let $R^\circ_{v,w} = (BwB \cap B_-vB)/B \subset \mathrm{Fl}(n)$ be the corresponding Richardson variety. Then we have bigraded isomorphisms*

$$H^*_{T,c}(R^\circ_{v,w}) \cong \mathrm{HHH}^0(F^\bullet_\beta)$$
$$H^*(R^\circ_{v,w}) \cong \mathrm{HHH}^0_{\mathbb{C}}(F^\bullet_\beta),$$

where $\mathrm{HHH}^0(-) := H^\bullet(\mathrm{HH}^0(-))$ is zero-th KhR homology and $\mathrm{HHH}^0_{\mathbb{C}}(-) := H^\bullet(\mathrm{HH}^0(-) \otimes_R \mathbb{C})$.

(The bigrading on honest cohomology comes from Hodge theory; more on that later.)

Example 1.1.2 (KhR helps Richardson)**.** Let $w_k = (k+1)\dots n1\dots k$. The Richardson braid $\beta_{\mathrm{id},w}$ is the $(k, n-k)$ torus link. It follows that the Hilbert series $\mathcal{H}(H^*_{T,c}(R^\circ_{\mathrm{id},w_k}); q, t)$ is the Hilbert series of torus link homology.

Assume that $\gcd(k, n) = 1$. In this case, Mellit [Mel17] showed that, after a renormalization (see [GL20, pg 27] for the precise statement), the Hilbert series $\mathcal{H}(\mathrm{HHH}^0(F^\bullet_\beta); q, t)$ is the q, t-Catalan number

$$C_{k,n-k}(q,t) = \sum_P q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}.$$

In particular, (after a renormalization) the Hilbert series $\mathcal{H}(H^*_{T,c}(R^\circ_{\mathrm{id},w_k}); q, t)$ is also equal to the q, t-Catalan number. I believe that this fact was not known before [GL20]. △

Remark 1.1.3. For any k, n, the Richardson variety $R^\circ_{\mathrm{id},w_k}$ is a friendly object: the projection $\mathrm{Fl}(n) \to \mathrm{Gr}_{k,n}$ gives an isomorphism from $R^\circ_{\mathrm{id},w_k}$ to the maximal open positroid variety $\Pi^\circ_{k,n} \subset \mathrm{Gr}_{k,n}$, that is,

$$R^\circ_{\mathrm{id},w_k} \cong \{V \in \mathrm{Gr}_{k,n} \colon \Delta_{\{a+1,a+2,\dots,a+k\}}(V) \neq 0 \text{ for all } a\} \qquad (\text{indices taken mod } n).$$

(See [GL20, Prop 4.3] or [KLS11, Thm 5.19].) △

Problem 1.1.4. *Find a more geometric proof of the fact that* $\mathcal{H}(H^*_{T,c}(R^\circ_{\mathrm{id},w_k}); q, t) \approx C_{k,n-k}(q, t)$.

Remark 1.1.5. Galashin and Lam make sure of intermediate renormalizations $\mathcal{P}_{\mathrm{KR}}$ and $\mathcal{P}(Y)$ given in [GL20, (3.15)], [GL20, (4.6)]. These are not the same as $\mathcal{H}$ above. △

Example 1.1.6 (Richardson helps KhR)**.** It is known that every open positroid variety [GL19] (or more generally, I think, any open Richardson variety [GLSBS22]) is a locally acyclic cluster variety. The cohomology of these varieties are of *Hodge-Tate type*, meaning that $H^{n,(p,q)}(X, \mathbb{C}) \neq 0$ only when $p = q$, and the varieties also satisfy *curious Lefschetz*, meaning that there exists $\gamma \in H^{2,(2,2)}(X, \mathbb{C})$ so that

$$\gamma^{d-p} \colon H^{p+s,(p,p)}(X, \mathbb{C}) \xrightarrow{\sim} H^{2d-p+s,(2d-p,2d-p)}(X, \mathbb{C}).$$

As with usual Lefschetz, these isomorphisms imply that certain dimension counts $h^{n,p} := \dim H^{n,(p,p)}$ are unimodal and symmetric:

$$h^{s,0} \leq h^{2+s,2} \leq \dots \leq h^{d+s,d} \geq \dots \geq h^{2d-2,2d-s-2} \geq h^{2d,2d-s},$$

$$h^{1+s,1} \leq h^{3+s,3} \leq \dots \leq h^{d-1+s,d-1} = h^{d+1+s,d+1} \geq \dots \geq h^{2d-3,2d-s-3} \geq h^{2d-1,2d-s-1}.$$

These properties imply that (after renormalization), the Hilbert series $\mathcal{H}(\mathrm{HHH}^0(F^\bullet_\beta); q, t)$ is q, t-symmetric and q, t-unimodal in the sense that for any d the coefficients of $q^d, q^{d-1}t, q^{d-2}t^2, \dots, t^d$ are unimodal. △

Remark 1.1.7. There is a direct proof ([GL20, Thm 4.11]) that $H^*(R^\circ_{v,w}, \mathbb{C})$ and $H^*_{T,c}(R^\circ_{v,w}, \mathbb{C})$ are Hodge-Tate, but they don't prove curious Lefschetz.

It is known ([Bri04, pg 12], [GL20, Lem 4.4]) that $R^\circ_{v,w}$ is a smooth affine variety. Smoothness implies that the Hodge weights of $H^k(R^\circ_{v,w}, \mathbb{C})$ are concentrated in degrees $\{k, k+1, \dots, 2k\}$. △

1.2 Conjectures and other results

When $wv^{-1} \in S_n$ is a single cycle, the torus T acts freely on $R^\circ_{v,w}$ and the quotient $R^\circ_{v,w} \to R^\circ_{v,w}/T$ has a section. (This automatically implies that $R^\circ_{v,w} \cong (R^\circ_{v,w}/T) \times T$.)

When $\gcd(k, n) = 1$, the permutation $w_k \in S_n$ is a single cycle and the variety $\mathcal{X}^\circ_{k,n} := \Pi^\circ_{k,n}/T$ is known to be a smooth affine cluster variety.

Conjecture 1.2.1 ([GL20, Conj 1.21])**.** *There is a deformation retraction from $\mathcal{X}^\circ_{k,n}$ to the compactified Jacobian $J_{k,n-k}$ of the plane curve singularity $x^k = y^{n-k}$ sending the weight filtration of $H^*(\mathcal{X}^\circ_{k,n})$ to the perverse filtration of $H^*(J_{k,n-k})$.*

[$\star$Is $H^*(J_{k,n-k})$ already known to be the q,t-Catalan? [$\star$Yes? I forgot, sorry...$\star$]$\star$]

Hogancamp and his coauthors have found recurrences ([Hog17] for various products of full twists and Jucys-Murphy braids, [HM19] for torus links, and [EH16] for some other stuff(?)) which compute $\mathrm{HHH}^0(F^\bullet_\beta)$ for various links β.

Problem 1.2.2. *Find analogous recurrences for* $\mathrm{HHH}^0(F^\bullet_\beta)$ *and* $\mathrm{HHH}^0_{\mathbb{C}}(F^\bullet_\beta)$ *for Richardson links* β*.*

Remark 1.2.3. These recurrences involve Rouquier complexes which are not $F^\bullet_\beta$ for any torus link. However, all involved Rouquier complexes have vanishing odd cohomology, and this property is useful for Hogancamp and his coauthors.

It is known [GL20, Ex 4.21–4.23] that not all Richardson links have vanishing odd cohomology. Perhaps one should restrict to special classes of Richardson links. △

There are standard techniques [LS16, LS21] to compute the cohomology of cluster varieties by relating them to smaller cluster varieties via the long exact sequence. These give recurrences for positroid varieties [GL22, (5.2)].

Problem 1.2.4. *Relate these recurrences to Hogancamp's recurrences.*

Lefschetz theorems often lead to symmetry and unimodality; combinatorialists love this. Combinatorialists also love log-concavity, and these theorems often follow from Hodge-Riemann relations.

Problem 1.2.5. *Is there a "curious Hodge-Riemann" relation for* $\mathrm{HHH}^0(F^\bullet_\beta)$*, and does it imply log-concavity?*

It is known that $\mathrm{HHH}^0(F^\bullet_\beta)$ is (essentially) a link invariant. Xinchun and I have some small progress on the following general problem:

Problem 1.2.6. *Find links which are closures of different Richardson braids, and use link invariance of* HHH^0 *to relate cohomologies of disparate Richardson varieties or compute cohomology of more Richardson varieties.*

There is a y-ified version $\mathrm{HY}(F^\bullet_\beta)$ of KhR homology introduced by Gorsky and Hogancamp [GH17]. In some sense it is supposed to play the role of equivariant KhR; for example [GH17, Thm 1.17] feels like the localization theorem in equivariant cohomology: in many cases $\mathrm{HY}(F^\bullet_\beta)$ is a free module and the map into $\mathrm{HY}(F^\bullet_{\mathrm{split}(\beta)})$ is an injection.

Problem 1.2.7. *Is there an analogue of* [GL20] *for* $\mathrm{HY}(F^\bullet_\beta)$*?*

2 Cluster algebras

[$\star E_6$-singularity: $H_1(F)$ of Milnor fiber has E_6 intersection pattern$\star$]

[$\star P = W?\star$]

The eventual goal of the next two talks is to discuss Lam and Speyer's paper "Cohomology of cluster varieties I. Locally acyclic case" [LS16]. I want to state curious Lefschetz precisely and discuss some surroundings (e.g. how to build curious Lefschetz for large varieties out of smaller ones).

In this talk I discuss the basics of cluster algebras. I want to highlight some beautiful, if maybe irrelevant, bits of the theory as well. I will not be able to do the subject justice. The theory began in a series of four papers [FZ01, FZ02, BFZ03, FZ06] called "Cluster Algebras I–IV" along with many auxillary papers. There is a textbook called "Introduction to cluster algebras" in preparation; preliminary drafts are on the arXiv [FWZ16, FWZ17, FWZ20, FWZ21]. Fomin has a Proc. ICM survey [Fom10], as well.

2.1 Cluster algebra preamble

Definition 2.1.1. An *ice quiver* is a quiver which $n+m$ vertices where:

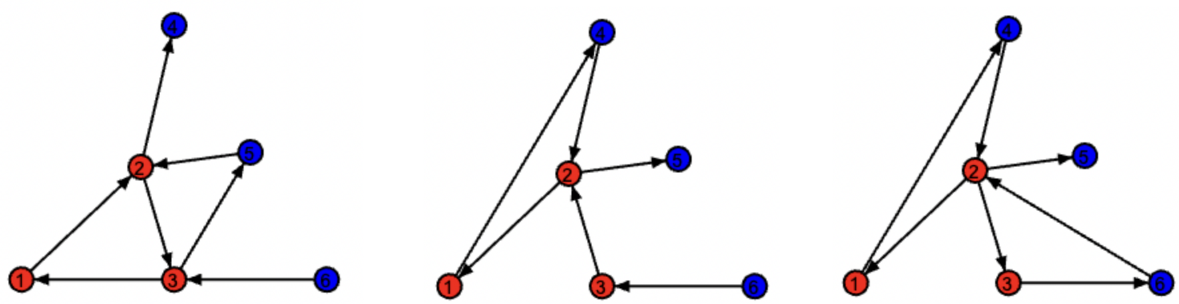
- m of the vertices are designated *frozen* and the other n vertices are designated *mutable*,

- there are no directed edges between frozen vertices,
- there are no loops or directed 2-cycles. △

Definition 2.1.2. Let Q be an ice quiver. For any mutable vertex k, the mutated quiver $\mu_k(Q)$ is obtained from Q by modifying its edges as follows:

- For every $i \to k \to j$ such that at least one of $\{i, j\}$ is mutable, add an arrow $i \to j$
- Reverse all arrows $i \to k$ and $k \to j$
- Cancel out any 2-cycles. △

Example 2.1.3. The quiver on the left in the figure below is an ice quiver Q with mutable vertices $\{2, 4, 5\}$ and frozen vertices $\{1, 3, 6\}$. The quiver in the middle is the ice quiver $\mu_2(Q)$. The quiver on the right is the ice quiver $\mu_3(\mu_2(Q))$.



△

Fix integers $n, m \geq 0$ and let $x_1, \ldots, x_{n+m}$ denote indeterminates. (Later, we will define the cluster algebra associated to an ice quiver with n mutable vertices and m frozen vertices.)

Definition 2.1.4. An *extended cluster* is a tuple $\mathbf{x} = (\varphi_1, \ldots, \varphi_{n+m})$ of rational functions in the x_i. The rational functions $\varphi_1, \ldots, \varphi_n$ are called *cluster variables* and the rational functions $\varphi_{n+1}, \ldots, \varphi_{n+m}$ are called *frozen variables*. △

Definition 2.1.5. A *seed* is a pair $(\mathbf{x}, Q)$ where $\mathbf{x}$ is an extended cluster and Q is an ice quiver. △

Definition 2.1.6. Let $(\mathbf{x}, Q)$ be a seed and let k be a mutable vertex of Q. The mutated seed $\mu_k(\mathbf{x}, Q)$ is the seed $(\mathbf{x}', \mu_k(Q))$ where $\mathbf{x}'$ is obtained by replacing φ_k with

$$\varphi'_k = \left(\prod_{\substack{e \in E \\ s(e)=k}} \varphi_{t(e)} + \prod_{\substack{e \in E \\ t(e)=k}} \varphi_{s(e)} \right) \cdot \frac{1}{\varphi_k}.$$

The relation

$$\varphi'_k \varphi_k = \prod_{\substack{e \in E \\ s(e)=k}} \varphi_{t(e)} + \prod_{\substack{e \in E \\ t(e)=k}} \varphi_{s(e)}$$

is called the *exchange relation*. △

Example 2.1.7. Let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ and let Q be as in Example 2.1.3. The seed $\mu_2(\mathbf{x}, Q)$ is $(\mathbf{x}', \mu_2(Q))$ where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by replacing x_2 with

$$x'_2 := \frac{x_3 x_4 + x_1 x_5}{x_2}.$$

The seed $\mu_3(\mu_2(\mathbf{x}, Q))$ is $(\mathbf{x}'', \mu_3(\mu_2(Q)))$ where $\mathbf{x}''$ is obtained from $\mathbf{x}'$ by replacing x_3 with

$$x_3' := \frac{x_2' + x_6}{x_3} = \frac{x_3x_4 + x_1x_5 + x_6}{x_2x_3}.$$

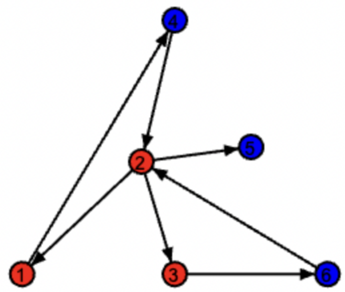
So, to summarize, we have

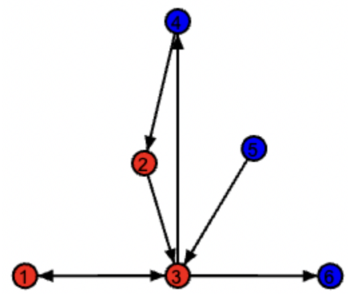
$$\mathbf{x}'' = \left(x_1, \frac{x_3x_4 + x_1x_5}{x_2}, \frac{x_3x_4 + x_1x_5 + x_6}{x_2x_3}, x_4, x_5, x_6\right).$$

△

Lemma 2.1.8. *Mutation of seeds is an involution:* $\mu_k(\mu_k(\mathbf{x}, Q)) = (\mathbf{x}, Q)$.

Remark 2.1.9. Mutations do not commute: $\mu_k(\mu_\ell(\mathbf{x}, Q))$ need not be equal to $\mu_\ell(\mu_k(\mathbf{x}, Q))$. They *do* commute if $\#\{k \to \ell\} = \#\{\ell \to k\} = 0$. Even quiver mutation does not commute: below is $\mu_3(\mu_2(Q))$ and $\mu_2(\mu_3(Q))$, where Q is as in Example 2.1.3:





△

2.2 Cluster algebras from ice quivers

In what follows, fix an ice quiver Q with n mutable vertices and m frozen vertices.

Definition 2.2.1. The *initial seed* is the pair $(\mathbf{x}_\emptyset, Q_\emptyset)$ where $\mathbf{x}_\emptyset = (x_1, \ldots, x_{n+m})$ and $Q_\emptyset = Q$. △

Definition 2.2.2. For a finite sequence $\mathbf{t} = (t_1, t_2, \ldots, t_\ell)$ of mutable vertices $t_i \in [n]$, define $(\mathbf{x}_\mathbf{t}, Q_t)$ to be the seed $\mu_{t_\ell}(\mu_{t_{\ell-1}}(\ldots(\mu_{t_1}(\mathbf{x}_\emptyset, Q_\emptyset))\ldots))$. Denote the cluster variables in $\mathbf{x}_\mathbf{t}$ by $\varphi_{\mathbf{t};i}(x_1, \ldots, x_{n+m})$. △

Definition 2.2.3. The cluster algebra $\mathcal{A}$ associated to Q is any of:

(I) $R[\varphi_{\mathbf{t};i}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})]$, where $R = \mathbb{C}[x_{n+1}, \ldots, x_{n+m}]$,

(II) $R'[\varphi_{\mathbf{t};i}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})]$, where $R' = \mathbb{C}[x_{n+1}^\pm, \ldots, x_{n+m}^\pm]$,

(III) $\mathbb{C}[\varphi_{\mathbf{t};i}(x_1, \ldots, x_n, 1, \ldots, 1)]$.

△

Remark 2.2.4.

- The convention in [LS16] will be to use (II). We will see that inverting the frozen variables amounts to restricting from a Grassmannian to the maximal open positroid. (You can imagine, though, that much of the theory is focused on (I) or (III).)
- The construction (III) can actually be viewed as "construction (I) or (II) for the full subquiver of Q on mutable vertices". (This is [Mul11, Prop 3.7].)

△

Example 2.2.5. Let Q be the type A_2 Dynkin quiver. Let's compute the cluster algebra $\mathcal{A}$ associated to Q.

In this case, quiver mutation at either mutable vertex amounts to flipping the orientation of the edge. Let us write Q' for the quiver Q with the edge reversed.

The exchange relation will always be of the form

$$\varphi_k' = \frac{\varphi_\ell + 1}{\varphi_k}, \qquad \{k, \ell\} = \{1, 2\}.$$

So we compute:

$$\mu_1((x_1, x_2), Q) = \left(\left(\frac{x_2+1}{x_1}, x_2\right), Q'\right)$$

$$\begin{aligned}\mu_2\mu_1((x_1, x_2), Q) &= \left(\left(\frac{x_2+1}{x_1}, \frac{\frac{x_2+1}{x_1}+1}{x_2}\right), Q\right)\\ &= \left(\left(\frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, Q\right)\right)\end{aligned}$$

$$\begin{aligned}\mu_1\mu_2\mu_1((x_1, x_2), Q) &= \left(\left(\frac{\frac{x_1+x_2+1}{x_1x_2}+1}{\frac{x_2+1}{x_1}}, \frac{x_1+x_2+1}{x_1x_2}\right), Q'\right)\\ &= \left(\left(\frac{x_1+x_2+1+x_1x_2}{x_2(x_2+1)}, \frac{x_1+x_2+1}{x_1x_2}\right), Q'\right)\\ &\overset{(!!!)}{=} \left(\left(\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\right), Q'\right)\end{aligned}$$

$$\begin{aligned}\mu_2\mu_1\mu_2\mu_1((x_1, x_2), Q) &= \left(\left(\frac{x_1+1}{x_2}, \frac{\frac{x_1+1}{x_2}+1}{\frac{x_1+x_2+1}{x_1x_2}}\right), Q\right)\\ &\overset{(!!)}{=} \left(\left(\frac{x_1+1}{x_2}, x_1\right), Q\right)\end{aligned}$$

$$\begin{aligned}\mu_1\mu_2\mu_1\mu_2\mu_1((x_1, x_2), Q) &= \left(\left(\frac{x_1+1}{\frac{x_1+1}{x_2}}, x_1\right), Q'\right)\\ &\overset{(!)}{=} (x_2, x_1, Q').\end{aligned}$$

At this point we observe that stacking more $\ldots \mu_1\mu_2$'s on the seed will not produce any new rational functions; for example the cluster variables in $\mu_2\mu_1\mu_2\mu_1\mu_2\mu_1$ will be the same as those of μ_1. Similarly, one can check that the cluster variables arising from first applying μ_2 and then alternating mutations will not produce any new cluster variables.

We deduce that

$$\begin{aligned}\mathcal{A} &= \mathbb{C}\left[x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\right]\\ &= \mathbb{C}[a, b, c, d, e]/\langle ca - b - 1, db - a - 1, eab - a - b - 1\rangle\end{aligned}$$

△

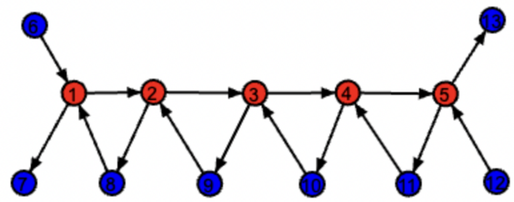
3 Cluster theory

Originally I intended to stay focused on things closely related to cluster varieties and link homology, but I got distracted and today I want to talk about some beautiful aspects of cluster theory.

[⋆Yixuan asked whether a cluster algebra is the affinization of the union of cluster tori. The answer is no in general, but yes for coordinate rings of open positroids. More on this later⋆]

3.1 Examples of cluster algebras

Proposition 3.1.1. *Let Q be the appropriate generalization of the ice quiver*



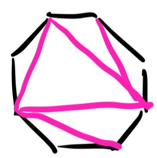
where one has an A_n Dynkin mutable quiver along with $n+3$ frozen vertices oriented as above.

Then the cluster algebra associated to Q has only finitely many cluster variables $\{\varphi_{\mathbf{t};i}\}$.

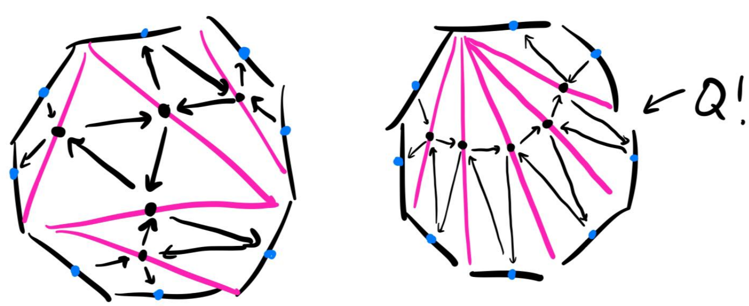
Remark 3.1.2. In particular, the cluster algebra associated to the type A_n Dynkin quiver with edges oriented "left to right" also has only finitely many cluster variables, as they are obtained by specializing frozen variables above to 1. See [Mul11, Prop 3.7] for a proof that deleting frozen vertices amounts to setting frozen variables equal to 1. △

Proof sketch. First I argue that $\{\mu_{\mathbf{t}}(Q)\}$ is finite.

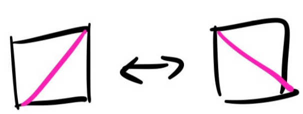
Consider a triangulated $(n+3)$-gon:



Given a triangulation T, construct the mutable quiver Q_T where the vertices correspond to edges of T and the edges correspond to pairs of edges of T which are part of a triangle, oriented clockwise:



A *diagonal flip* is:

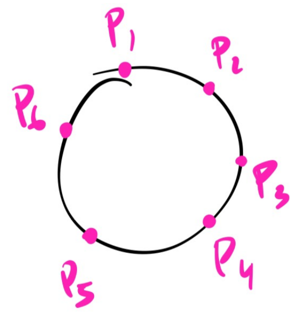


Lemma 3.1.3. *Let T' be the triangulation obtained from T by flipping the diagonal corresponding to the mutable vertex k of Q_T. Then $Q_{T'} = \mu_k(Q_T)$.*

It follows that the set of triangulations surjects onto the set $\{\mu_{\mathbf{t}}(Q)\}$.

Now I argue that $\{\varphi_{\mathbf{t};i}\}$ is finite. The strategy will be to show that for any fixed $\mathbf{x} = (x_1, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}_{>0}$, the set $\{\varphi_{\mathbf{t};i}(\mathbf{x})\}$ is finite. It then follows that only finitely many rational functions appear in the set $\{\varphi_{\mathbf{t};i}\}$.

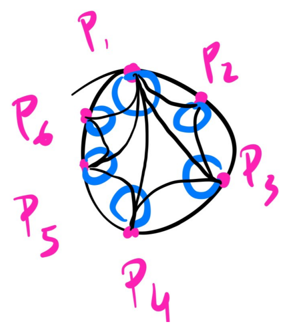
To this end I will introduce *Penner coordinates*. Fix $n+3$ points $p_1, \ldots, p_{n+3}$ on the unit circle.



Theorem 3.1.4 (Penner)**.** *Given $\mathbf{x} \in \mathbb{R}^{n+m}_{>0}$ there exists a hyperbolic metric on the unit disk and a collection of horocyles around each p_i so that*

$$x_i = \begin{cases} \lambda(p_1, p_{i+2}) & \text{when } i \leq n \\ \lambda(p_{i-n}, p_{i-n+1}) & \text{when } i \geq n+1 \end{cases}$$

where $\lambda(p_i, p_j) := \exp(\ell(p_i, p_j)/2)$ is the lambda length, and $\ell(p_i, p_j)$ is the length of the segment of the geodesic connecting p_i and p_j which is between the horocycles.



(I believe that this formulation of Penner's results is due to S. Fomin and D. Thurston; [FT12, Thm 7.4]. Penner proves earlier versions of this theorem in [Pen87, Thm 3.1] and [Pen04, Thm 5.10].)

The seed $(\mathbf{x}_\emptyset, Q_\emptyset)$ is thus encoded by a decorated hyperbolic metric g and the triangulation T.

Lemma 3.1.5. *For any decorated hyperbolic metric g and any 4-tuple of cyclically ordered points p_1, p_2, p_3, p_4 on the unit circle, the lambda lengths $\lambda(p_i, p_j)$ satisfy*

$$\lambda(p_1, p_3)\lambda(p_2, p_4) = \lambda(p_1, p_2)\lambda(p_3, p_4) + \lambda(p_2, p_3)\lambda(p_4, p_1).$$

Corollary 3.1.6. *Suppose that a seed $(\mathbf{x}, Q)$ is encoded by a decorated hyperbolic metric g and a triangulation T. Then, the mutated seed $\mu_k(\mathbf{x}, Q)$ is encoded by the decorated hyperbolic metric g and the triangulation T' obtained by flipping the diagonal corresponding to k.*

Thus, the set of lambda lengths $\{\lambda(p_i, p_j)\}$ surjects onto the set $\{\varphi_{\mathbf{t};i}(\mathbf{x})\}$; the result follows. □

Theorem 3.1.7 (cf. [FWZ20, Thm 6.7.8])**.** *Fix $k < n$. Let $\mathcal{A}$ be the cluster algebra of type (I), (II), (III) respectively associated to the ice quiver $Q_{k,n}$ given by (the appropriate generalization of) the ice quiver in Figure 1.*

Then $\mathcal{A}$ is the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$, $\Pi^\circ_{k,n}$, and $C_{k,n}$ respectively. [$\star$*The Catalan variety is only defined when* $\gcd(k,n) = 1$*, I believe*$\star$]

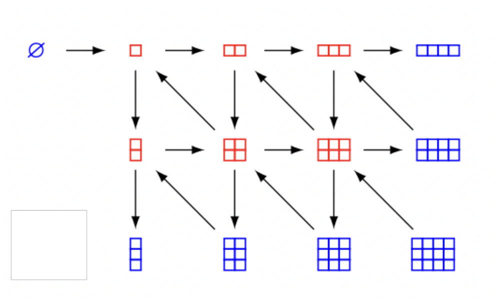


Figure 1: The quiver $Q_{3,7}$. One draws a $k \times (n-k)$ rectangle plus one vertex at the northwest, then freezes the south and east sides.

Proof sketch in the case $k = 2$. In the case $k = 2$, the mutable part of the ice quiver $Q_{2,n}$ is a type A_{n-3} quiver. We recycle some ideas from the proof of Proposition 3.1.1. In the proof we showed that a cluster variable $\varphi_{\mathbf{t};i}$ is naturally identified with a pair of vertices p_i, p_j: there was some "lambda length" recipe and one checked that for any 4-tuple of cyclically ordered points p_i, p_j, p_k, p_ℓ the lambda lengths satisfy the exchange relation $\lambda(p_i, p_k)\lambda(p_j, p_\ell) = \lambda(p_i, p_j)\lambda(p_k, p_\ell) + \lambda(p_j, p_k)\lambda(p_\ell, p_i)$.

I remind you that the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(2, n)$ is generated by the Plücker coordinates $\{\Delta_I \colon |I| = 2\}$ subject to the Plücker relations

$$\Delta_{\{i,j\}}\Delta_{\{k,\ell\}} - \Delta_{\{i,k\}}\Delta_{\{j,\ell\}} + \Delta_{\{i,\ell\}}\Delta_{\{j,k\}} = 0. \qquad \text{(for any } i\text{, and any } j < k < \ell\text{)}.$$

An isomorphism between $\mathcal{A}$ and $\mathbb{C}[\mathrm{Gr}(2, n)]$ is given by sending a cluster variable $\varphi_{\mathbf{t};i}$ corresponding to the lambda length $\lambda(p_i, p_j)$ to the Plücker coordinates $\Delta_{\{i,j\}}$.

To make all the details work (e.g. we should check that the ideal of relations among cluster variables is generated by exchange relations – which is not always true for general cluster algebras!), one can use [FWZ20, Prop 6.2.1]; it gives sufficient conditions to identify a given cluster algebra with a given coordinate ring.

Since the frozen variables correspond to the cyclic Plücker coordinates, we deduce that the cluster algebra of type (II) is the homogeneous coordinate ring of the maximal open positroid.

By [Mul11, Prop 3.7], deleting frozen vertices in a quiver amounts to setting frozen variables equal to 1 in the cluster algebra. Hence the cluster algebra of type (III) is the coordinate ring of the Catalan variety when $\gcd(k, n) = 1$. □

3.2 Finite type classification

Theorem 3.2.1 (Special case of [FZ02, Thm 1.8])**.** *Let Q be an ice quiver. The associated cluster algebra has finitely many cluster variables if and only if $\{\mu_{\mathbf{t}}(Q)\}$ contains an orientation of an ADE quiver.*

(Such cluster algebras are called *cluster algebras of finite type*.)

Remark 3.2.2. The full theorem [FZ02, Thm 1.8] goes something like this (see also [FWZ17, Ch 5]). One can generalize our ice quiver setup as follows: quivers give rise to skew-symmetric matrices $B = (b_{ij})$ encoding edges. Quiver and seed mutation are replaced with formulas in terms of the b_{ij} rather than in terms of edges of the quiver. We get a notion of a cluster algebra associated to a matrix and our quiver setup corresponded to those cluster algebras associated to skew-symmetric matrices.

Wisdom says that cluster algebras associated to skew-symmetrizable matrices are nearly as well-behaved as those associated to skew-symmetric matrices. Write $A(B) = (a_{ij})$ be the matrix

$$a_{ij} := \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j \end{cases}.$$

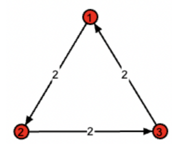
Then, [FZ02, Thm 1.8] (see also [FWZ17, Thm 5.2.8, Thm 5.2.11]) asserts that the cluster algebra associated to B has finite type if and only if it is mutation equivalent to a matrix B' so that $A(B')$ is a Cartan matrix of finite type (i.e. A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2). △

Remark 3.2.3. In [FWZ17, pg. 19], they explicitly say that they are not aware of a simple argument that would directly derive the classification from one of the instances of the classical Cartan-Killing classification. △

Remark 3.2.4. There is a classification of quivers for which $\{\mu_{\mathbf{t}}(Q)\}$ is finite due to Felikson, Shapiro, and Tumarkin [FST08, Thm 6.1]: Besides cluster algebras of rank 2 and cluster algebras associated with triangulated surfaces, there are exactly 11 exceptional cases of which 9 are root-system-ish and 2 seem completely random to me. △

3.3 Laurent phenomenon

Remark 3.3.1. Cluster algebras are not always finitely generated. I believe (never actually checked / saw a proof) that the algebra associated to



is not Noetherian because the Zariski tangent space at the maximal ideal generated by all cluster variables is infinite dimensional.

More generally, it is known [FWZ20, Prop 6.8.1] that a cluster algebra associated to a quiver Q with 3 mutable vertices is finitely generated if and only if it is mutation equivalent to an acyclic quiver. △

Theorem 3.3.2 ([FZ01, Thm 3.1])**.** *Fix any sequence* $\mathbf{s} = (s_1, \ldots, s_k)$*. Then every cluster variable* $\varphi_{\mathbf{t};i}$ *can be expressed as a Laurent polynomial in the quantities* $\{\varphi_{\mathbf{s};i}\}$*, that is to say,* $\mathcal{A} \subseteq \mathbb{C}[\varphi^{\pm}_{\mathbf{s};i}]$.

Example 3.3.3. We saw that the cluster variables for the A_2 quiver were

$$\left\{x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2}\right\}.$$

Evidently, all five variables are Laurent polynomials in the initial variables (x_1, x_2). But each variable is also a Laurent polynomial in the quantities $(\varphi_1, \varphi_2) := (\frac{x_1+1}{x_2}, \frac{x_1+x_2+1}{x_1x_2})$; for example, we have $x_2 = \frac{\varphi_1+\varphi_2+1}{\varphi_1\varphi_2}$. △

Definition 3.3.4. The *upper cluster algebra* associated to Q is the ring

$$\mathcal{U} := \bigcap_{\mathbf{t}} \mathbb{C}[\varphi^{\pm}_{\mathbf{t};i}]. \qquad \triangle$$

The Laurent phenomenon guarantees that $\mathcal{A} \subseteq \mathcal{U}$. Equality does not always hold, but it is known to hold in many good cases.

Definition 3.3.5. For any sequence $\mathbf{s} = (s_1, \ldots, s_k)$, the *cluster torus* is the image of $\mathrm{Spec}(\mathbb{C}[\varphi^{\pm}_{\mathbf{s};i}])$ in $\mathrm{Spec}(\mathcal{U})$ or $\mathrm{Spec}(\mathcal{A})$.

The union of the cluster tori is called the *cluster manifold* $\mathcal{X}^\circ$. △

Proposition 3.3.6 ([Mul11, Prop 2.6])**.** *The affine closure of* $\mathcal{X}^\circ$ *is* $\mathcal{U}$.

4 Cluster varieties and their cohomology

4.1 Louise property

Let $\mathcal{A}$ be a cluster algebra and let $(\mathbf{x}, Q)$ be a seed of $\mathcal{A}$. Let Q' be the quiver obtained from Q by freezing some mutable vertices $i_1, \ldots, i_s$, and let $\mathcal{A}'$ be the cluster algebra with initial seed $(\mathbf{x}, Q')$. (Note that $\mathbf{x} = (\varphi_1, \ldots, \varphi_{n+m})$ is some strange collection of rational functions in the x_i.)

Definition 4.1.1. We say $\mathcal{A}'$ is a *cluster localization* of $\mathcal{A}$ if $\mathcal{A} \subset \mathcal{A}'$ (as subalgebras of $\mathbb{Q}(x_1, \ldots, x_{n+m})$), or equivalently if $\mathcal{A}' = \mathcal{A}[\varphi_{i_1}^{-1}, \ldots, \varphi_{i_s}^{-1}]$.

The corresponding open subvariety $\mathrm{Spec}(\mathcal{A}') \subseteq \mathrm{Spec}(\mathcal{A})$ is called a *cluster chart*. △

Remark 4.1.2. See [Mul11, Prop 11.12] for an example of a freezing which is not a cluster localization. They find an ice quiver Q where $\mathcal{A}$ is finitely generated. However, the cluster algebra $\mathcal{A}^\dagger$ associated to the quiver $Q^\dagger$ given by deleting a vertex i is not finitely generated. Thus, if the cluster algebra $\mathcal{A}'$ associated to the quiver Q' given by freezing the vertex i was a cluster localization, then we have $\mathcal{A}^\dagger = \mathcal{A}'/\langle \varphi_i - 1 \rangle$, a contradiction. △

Definition 4.1.3. A cluster variety $\mathrm{Spec}(\mathcal{A})$ is called *locally acyclic* if it has a finite cover by acyclic cluster charts. △

Theorem 4.1.4 ([Mul11, Thm 4.1, Thm 4.2])**.** *Let* $\mathrm{Spec}(\mathcal{A})$ *be a locally acyclic cluster variety. Then* $\mathcal{A} = \mathcal{U}$*, and* $\mathcal{A}$ *is finitely generated, integrally closed, and locally a complete intersection.*

In what follows, it will be useful to encode the ice quiver Q into an $(n+m) \times n$ matrix $B(Q)$ whose entries are $b_{ij} = \ell$ if $\#\{i \to j\} = \ell > 0$ and $b_{ij} = -\ell$ if $\#\{j \to i\} = \ell > 0$ and $b_{ij} = 0$ otherwise. We say that $\mathcal{A}$ has full rank if the matrix $B(Q)$ has full rank.

Theorem 4.1.5 ([Mul11, Thm 7.7])**.** *If* $\mathcal{A}$ *is a locally cyclic cluster algebra of full rank, then* $\mathcal{A}$ *is regular.*

From now on, all cluster algebras will be with frozen variables inverted, i.e., "type (II) or (III)".

Definition 4.1.6. An edge e of a quiver is a *separating edge* if there is no bi-infinite path through e. △

Definition 4.1.7. A cluster algebra $\mathcal{A}$ satisfies the *Louise property* if either:

- For some seed $(\mathbf{x}, Q)$ of $\mathcal{A}$, the quiver Q is acyclic, or
- For some seed $(\mathbf{x}, Q)$ of $\mathcal{A}$, there exists a separating edge $i \to j$ of Q so that $\mathcal{A}(Q \setminus \{i\})$, $\mathcal{A}(Q \setminus \{j\})$, and $\mathcal{A}(Q \setminus \{i, j\})$ all satisfy the Louise property. △

Proposition 4.1.8 ([LS16, pg. 15])**.** *If the Louise property holds for* $\mathcal{A}$*, then* $\mathcal{A}$ *is locally acyclic.*

See [LS16, §4] for more details; as far as I can tell the key ingredient is to show that if $i \to j$ is separating then $\mathrm{Spec}\,\mathcal{A}$ is covered by the open subsets $\mathrm{Spec}(\mathcal{A}[\varphi_i^{-1}])$ and $\mathrm{Spec}(\mathcal{A}[\varphi_j^{-1}])$. Then one can use the fact ([Mul11, Lem 3.4, Thm 4.1]) that if a freezing $\mathcal{A}'$ of $\mathcal{A}$ is locally acyclic, then $\mathcal{A}'$ is a cluster localization.

Theorem 4.1.9 ([MS14, Thm 3.3] plus [GL19, Thm 3.5, Prop 4.9])**.** *The coordinate ring* $\mathbb{C}[\Pi_f^\circ]$ *is a Louise cluster algebra of full rank.*

4.2 Curious Lefschetz for cluster varieties

Recall that complex algebraic varieties come equipped with a *mixed Hodge structure*: there is a Deligne splitting $H^k(X, \mathbb{C}) = \oplus_{p,q} H^{k,(p,q)}(X)$. We say $H^*(X)$ is of *Hodge-Tate type* if $H^{k,(p,q)}(X) = 0$ unless $p = q$, and we say $H^*(X)$ is *split over* $\mathbb{Q}$ if each summand $H^{k,(p,q)}(X)$ has a basis coming from $H^*(X, \mathbb{Q})$.

Let X be an even-dimensional affine variety and let $[\gamma] \in H^{2,(2,2)}(X, \mathbb{C})$. We say that $(X, [\gamma])$ satisfies *curious Lefschetz* if $H^*(X, \mathbb{C})$ is of Hodge-Tate type and if for all $s \geq 0$ and $p \leq d$, we have

$$\gamma^{d-p} \colon H^{p+s,(p,p)}(X, \mathbb{C}) \xrightarrow{\sim} H^{2d-p+s,(2d-p,2d-p)}(X, \mathbb{C}).$$

It is known [LS16, Thm 3.3] that if X satisfies the curious Lefschetz property then its cohomology is split over $\mathbb{Q}$.

Given an ice quiver Q, define a *Gekhtman–Shapiro–Vainshtein form* (or *GSV form*) to be any 2-form

$$\gamma = \sum_{i,j=1}^{n+m} \hat{B}_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}$$

for any $(n+m) \times (n+m)$ skew symmetric matrix $\hat{B}$ extending $B(Q)$.

The main theorems in [LS16] are as follows.

Theorem 4.2.1 ([LS16, §1.2])**.**

- *Suppose that $\mathcal{A}$ is Louise and full rank. Then $\mathcal{A}$ is smooth and the mixed Hodge structure of $H^*(\mathcal{A}, \mathbb{C})$ is of Hodge-Tate type and is split over $\mathbb{Q}$.*
- *Suppose that $\mathcal{A}$ is even-dimensional, Louise, and full rank, and let γ be any GSV-form. Then $(\mathcal{A}, [\gamma])$ satisfies the curious Lefschetz property.*
- *Suppose that $\mathcal{A}$ is Louise and full rank. For $e := \dim(\mathcal{A})$, we have $\dim H^{p+s,(p,p)}(\mathcal{A}, \mathbb{C}) = \dim H^{e-p+s,(e-p,e-p)}(\mathcal{A}, \mathbb{C})$.*

Proof sketch. The key ingredient is [LS16, Thm 3.5]: if X is a $2d$-dimensional smooth affine variety and $\gamma \in H^{2,(2,2)}(X)$, and if U and V are open affine subvarieties so that (U, γ), (V, γ), and $(U \cap V, \gamma)$ all satisfies curious Lefschetz, then so does (X, γ). The proof of this ingredient is basically Mayer-Vietoris and the five lemma.

We will do an induction. The base case is done by hand [LS16, Prop 8.2]; general theory [Mul11, Thm 6.5] says it suffices to consider isolated cluster algebras rather than acyclic ones in general.

Now, given an even-dimensional, Louise, full rank $\mathcal{A}$, we can find a seed $(\mathbf{x}, Q)$ with a separating edge $i \to j$. We know that $U := \mathrm{Spec}(\mathcal{A}[\varphi_i^{-1}])$ and $V := \mathrm{Spec}(\mathcal{A}[\varphi_j^{-1}])$ and $U \cap V := \mathrm{Spec}(\mathcal{A}[\varphi_i^{-1}, \varphi_j^{-1}])$ are even-dimensional, Louise, full rank, and have fewer mutable vertices; furthermore, the restrictions of any GSV form γ to these open subvarieties is again a GSV form. By induction, (U, γ), (V, γ), and $(U \cap V, \gamma)$ all satisfy curious Lefschetz; the key ingredient [LS16, Thm 3.5] guarantees that (X, γ) satisfies curious Lefschetz. This proves all parts of the theorem in the even-dimensional case.

Given an odd-dimensional cluster variety $\mathcal{A}$ which is Louise and full rank, observe that $\mathcal{A} \times \mathbb{C}^\times$ is again a cluster variety which is Louise and full-rank. It is also even-dimensional, so using the result for $\mathcal{A} \times \mathbb{C}^\times$ and the fact that Künneth preserves the Deligne splitting gives the desired result. □

The last trick we used above has the following useful generalization:

Proposition 4.2.2 ([LS16, Prop 5.11])**.** *Let $\tilde{B}$ and $\tilde{B}'$ be $(n+m) \times n$ and $(n+m') \times n$ exchange matrices with the same top n rows and whose rows have the same integer span. Let $\mathcal{A}$ and $\mathcal{A}'$ be the corresponding cluster varieties. Then $\mathcal{A} \times (\mathbb{C}^\times)^{m'} \cong \mathcal{A}' \times (\mathbb{C}^\times)^m$. If $m = m'$, then we have natural isomorphism $H^{k,(p,q)}(\mathcal{A}) = H^{k,(p,q)}(\mathcal{A}')$.*

In light of this result we say that $\tilde{B}$ has *really full rank* if its integer span is $\mathbb{Z}^n$.

We remark that $\mathcal{A} \times \mathbb{C}^\times \cong \mathcal{A}' \times \mathbb{C}^\times$ does not imply $\mathcal{A} \cong \mathcal{A}'$ in general, but does imply they have the same Betti numbers. So the result says that you can add/remove frozen variables somewhat indiscriminately and the Betti numbers change predictably.

4.3 Examples and computation

Example 4.3.1. Let Q be a type A_{2n} Dynkin quiver with no frozen vertices and let $\mathcal{A}$ be the corresponding cluster variety. The corresponding matrix is really full rank and the Betti numbers are given in [LS16, §6.2] by the table

	H^0	H^1	H^2	H^3	$\dots$	H^{2n-1}	H^{2n}
$k-p=0$	1	0	1	0	$\dots$	0	1

Table 1: Type A_{2n}

We saw last time that the cluster algebra associated to Q is the Catalan variety $\mathcal{X}_{2,2n+3}$. Note that $\dim(\Pi^\circ_{2,2n+3}) = 2(2n+1)$ and $\dim(\mathcal{X}_{2,2n+3}) = 2n$. By Poincaré duality and the fact that T acts freely, we get

$$\begin{aligned} H^{k,(k,k)}(\mathcal{X}_{2,2n+3}) &\cong H_c^{4n-k,(2n-k,2n-k)}(\mathcal{X}_{2,2n+3}) \\ &\cong H_{T,c}^{4n-k+(n+2),(2n-k,2n-k)}(\Pi^\circ_{2,2n+3}). \end{aligned}$$

Galashin and Lam showed that $H^*_{T,c}(\Pi^\circ_{k,n})$ records the KR homology of the $(k, n-k)$ torus link, which in turn records the rational Catalan number $C_{k,n-k}$.

Thus, up to renormalization Table 1 records the the Hilbert series of the KR homology of the $(2, 2n+1)$ torus knot, i.e. of the rational Catalan number $C_{2,2n+1}(q,t) = q^n + q^{n-1}t + \cdots + t^n$. △

Example 4.3.2. It is known ([FWZ16, Ex 2.6.8]) that the quivers $Q_{k,n}$ defining small quivers are sometimes mutation-equivalent to acyclic quivers:

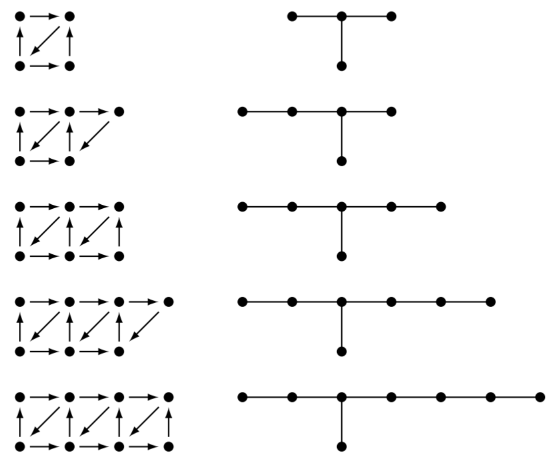


Figure 2.12. Quivers mutation equivalent to orientations of Dynkin diagrams of types D_4, D_5, E_6, E_7, E_8.

In particular, the Catalan varieties $\mathcal{X}^{\circ}_{3,7}$ and $\mathcal{X}^{\circ}_{3,8}$ (which are associated to the quivers $Q_{3,4}$ and $Q_{3,5}$ with frozen variables removed, respectively) are equal to the type E_6 and E_8 cluster algebras. For these cluster algebras, the matrices are really full rank and and the cohomology is computed as

	H^0	H^1	H^2	H^3	H^4	H^5	H^6
$k-p=0$	1	0	1	0	1	0	1
$k-p=1$					1		

Table 2: Type E_6

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	0	1		

Table 3: Type E_8

to be compared with the rational Catalan numbers

$$C_{3,4}(q,t) = (q^3 + q^2t + qt^2 + t^3) + qt$$
$$C_{3,5}(q,t) = (q^4 + q^3t + q^2t^2 + qt^3 + t^4) + (q^2t + qt^2),$$

cf. [GL20, Fig 1]. △

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