Rational Cherednik Algebras and Torus Knot Homology

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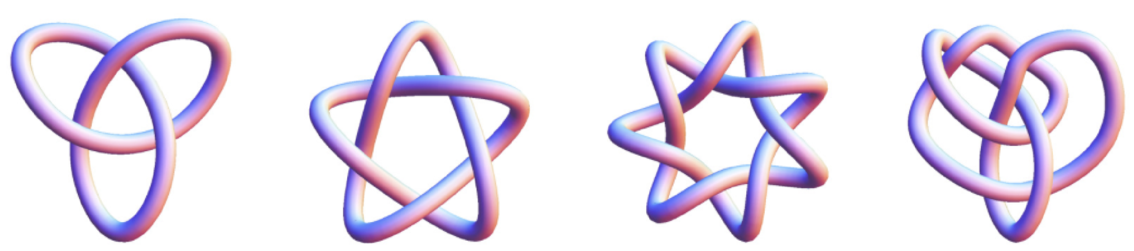


Definition of the RCA

Fix an integer $n \geq 2$. Let $G = \mathrm{SL}_n$ and $W = S_n$.
Fix the maximal torus $T \subset G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, consisting of diagonal matrices. For any $c \in \mathbb{C}$, define the rational Cherednik algebra $\mathrm{H}_c := \mathrm{H}_c(\mathfrak{h}, W)$ to be the $\mathbb{C}$-algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and W with relations

$$[x, x'] = [y, y'] = 0, \quad wxw^{-1} = w(x), \quad wyw^{-1} = w(y)$$
$$[y, x] = x(y) - \sum_{s \in S} c\langle \alpha_s, y\rangle\langle \alpha_s^\vee, x\rangle s$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $w \in W$, $S \subset W$ is the set of simple reflections and α_s, resp. $\alpha_s^\vee$, is the root, resp. coroot, associated to s. In particular $\mathrm{H}_0 = \mathcal{D}(\mathfrak{h}) \ltimes S_n$.

The finite-dimensional representation

For any W-representation τ, we may regard it as a $\mathbb{C}[\mathfrak{h}^*] \ltimes W$-representation by requiring $\mathbb{C}[\mathfrak{h}^*]$ to act trivially and define the Verma module $M_c(\tau) = \mathrm{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \ltimes W} \tau$.
The Verma module $M_c(\tau)$ has a maximal proper submodule with an irreducible quotient $\mathrm{L}_c(\tau)$. As τ varies over irreducible W-representations, $\mathrm{L}_c(\tau)$'s give a complete list of irreducible objects in $\mathcal{O}(\mathrm{H}_c)$. Write $\mathrm{L}_c := \mathrm{L}_c(\mathrm{triv})$.

Theorem([BEG03,Theorem 1.2]): When $c = \frac{m}{n}$ for positive integer m coprime to n, the only irreducible finite-dimensional representation of H_c is L_c. Moreover, only when $c = \frac{m}{n}$ for integer m coprime to n does H_c have finite-dimensional representations.

An example

When $n = 2$, H_c is generated by $x_1 - x_2$, $y_1 - y_2$ and $s = (12)$. We compute

$$(y_1 - y_2)(x_1 - x_2)^k = \left(\frac{\partial}{\partial_{x_1}} - \frac{\partial}{\partial_{x_2}} - 2c\frac{1-s}{x_1 - x_2}\right)(x_1 - x_2)^k = \begin{cases} (2k - 4c)(x_1 - x_2)^{k-1} & \text{when } k \text{ is odd} \\ 2k(x_1 - x_2)^{k-1} & \text{when } k \text{ is even} \end{cases}$$

As a result, for odd positive integer k we have a short exact sequence

$$0 \to ((x_1 - x_2)^k) \to \mathbb{C}[\mathfrak{h}] \to \mathrm{L}_{\frac{k}{2}} \to 0.$$

Shuffle generators

Define $K = \mathbb{C}(q,t)(z_1, z_2, \cdots)^{S_\infty}$. We endow K with a $\mathbb{C}(q,t)$-algebra structure via the shuffle product. The shuffle algebra $\mathfrak{A}$ is defined as a certain subspace of K.

- ([SV13]) There is an isomorphism between $\mathfrak{A}$ and the positive half of the elliptic Hall algebra.
- ([FT11, SV13]) There exists a geometric action of the algebra $\mathfrak{A}$ on $\oplus_{n \geq 0} K_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}^n) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(q,t)$.
- The shuffle algebra $\mathfrak{A}$ can be generated by

$$P_{n,m} = \mathrm{Sym}\left(\frac{\prod_{i=1}^n z_{n-i+1}^{\lfloor ic \rfloor - \lfloor (i-1)c \rfloor}}{\prod_{i=1}^{n-1}(1 - qt\frac{z_i}{z_{i+1}})} \prod_{1 \leq i < j \leq n} \omega\left(\frac{z_i}{z_j}\right)\right)$$

for $n \geq 1$, $m \in \mathbb{Z}$ and $\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}$.

Results

- **Theorem A**: When $m > n$ for coprime m, n, the bigraded Frobenius character of $\mathrm{L}_{\frac{m}{n}}$ with respect to the Hodge filtration F^{Hod} and the Euler field $\mathrm{h}_{\frac{m}{n}} := \Sigma(x_i y_i + y_i x_i)$ is given by

$$\mathrm{ch}_{S_n \times \mathbb{C}^* \times \mathbb{C}^*}(\mathrm{L}_{\frac{m}{n}}) = (P_{m,n} \cdot 1)(q, q^{-1}t).$$

  This confirms [GN15, Conjecture 5.5].

  Let $\mathrm{HHH}(T_{m,n})$ denote the Khovanov-Rozansky homology of the (m,n)-torus knot.

  Theorem([Mel22, Corollary 3.4]): Up to a constant normalization, the triply graded Euler characteristic $\mathrm{ch}_{a,q,t}(\mathrm{HHH}(T_{m,n}))$ equals $\langle \Lambda(\mathcal{V}, a) | P_{m,n} | 1 \rangle$.

- **Theorem B**: With respect to F^{Hod} and $\mathrm{h}_{\frac{m}{n}}$, when $m > n$ for coprime m, n there is a triply graded isomorphism

$$\mathrm{HHH}(T_{m,n}) \cong \mathrm{Hom}_{S_n}(\wedge^\bullet \mathfrak{h}, \mathrm{gr}^F_\bullet \oplus \mathrm{L}_{\frac{m}{n}}(\bullet)).$$

  where the following gradings are matched

  internal q-grading $\leftrightarrow$ $\mathrm{h}_{\frac{m}{n}}$-grading
  Hochschild homological a-grading $\leftrightarrow$ wedge degree of $\wedge^\bullet \mathfrak{h}$
  usual homological t-grading $\leftrightarrow$ filtration on $\mathrm{L}_{\frac{m}{n}}$

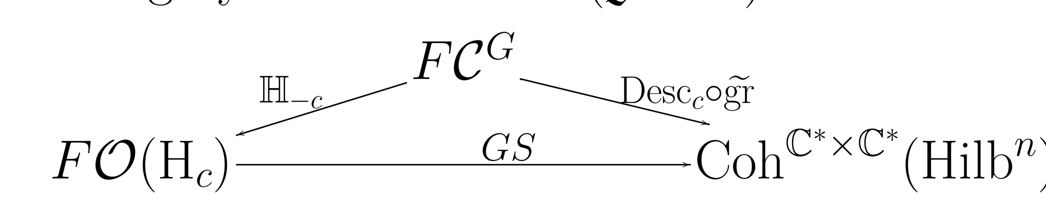
  This confirms [GORS14, Conjecture 1.2].

- **Theorem C**: $F^{\mathrm{Hod}} = F^{\mathrm{ind}} = F^{\mathrm{alg}}$ on $\mathrm{L}_{\frac{m}{n}}$ when $m > n$ for coprime m, n and on $(\mathrm{L}_{\frac{m}{n}})^{S_n}$ for all $m > 0$ coprime to n.

- **Corollary**: For all $m > 0$ coprime to n, with respect to the algebraic filtration, Theorem B holds.

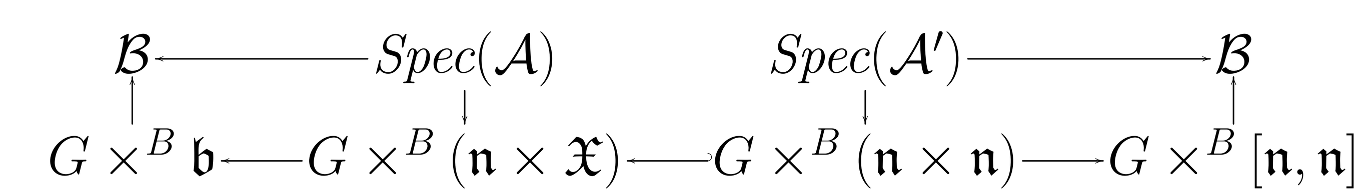
A diagram

Let $c = \frac{m}{n}$ and $V := \mathbb{C}^n$ be the standard representation of GL_n.
$F\mathcal{O}(\mathrm{H}_c)$: the category of filtered modules in $\mathcal{O}(\mathrm{H}_c)$
$F\mathcal{C}^G$: the category of mirabolic $\mathcal{D}(\mathfrak{g} \times V)$-modules.



$\mathbf{N}_c$: cuspidal mirabolic D-module.

Two dg modules



Define weights:
$\mu_{\lceil c \rceil} := (\lceil c \rceil, \lceil 2c \rceil - \lceil c \rceil, \ldots, \lceil nc \rceil - \lceil (n-1)c \rceil)$.
$\mu_{\lfloor c \rfloor} := (\lfloor c \rfloor, \lfloor 2c \rfloor - \lfloor c \rfloor, \ldots, \lfloor nc \rfloor - \lfloor (n-1)c \rfloor)$.
Cuspidal dg module of slope c:

$$\mathcal{A}_c := (\pi_{G \times^B (\mathfrak{n} \times \mathfrak{X}) \to \mathcal{B}})^* \mathcal{L}_{\mu_{\lceil c \rceil}} \otimes^L \mathcal{A}$$

Catalan dg module of slope c:

$$\mathcal{A}'_c := (\pi_{G \times^B (\mathfrak{n} \times \mathfrak{n}) \to \mathcal{B}})^* \mathcal{L}_{\mu_{\lfloor c \rfloor}} \otimes^L \mathcal{A}'$$

Theorem A is shown via the following:

- **Theorem**([CEE09]): $\mathbb{H}_{-c}(\mathbf{N}_c) \cong \mathrm{L}_c^{S_n}$.
  $\Rightarrow \mathrm{ch}_{\mathbb{C}^* \times \mathbb{C}^*}(\Gamma(\widetilde{\mathrm{gr}}^{\mathrm{Hod}} \mathbf{N}_c)^{\mathrm{GL}_n}) = \mathrm{ch}_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{L}_c^{S_n})$.
- Laumon's formula $\Rightarrow \widetilde{\mathrm{gr}}(\mathbf{N}_c) = Rp_* \mathcal{A}_c \boxtimes \mathcal{O}_V$
- A "shuffle argument" of Neguț $\Rightarrow$

$$\mathrm{ch}_{\mathbb{C}^* \times \mathbb{C}^*}(\Gamma(\mathcal{A}_c)^{\mathrm{GL}_n}) = \mathrm{ch}_{\mathbb{C}^* \times \mathbb{C}^*}(\Gamma(\mathcal{A}'_c)^{\mathrm{GL}_n}).$$

- Localization formula $\Rightarrow \mathrm{ch}_{\mathbb{C}^* \times \mathbb{C}^*}(\Gamma(\mathcal{P} \otimes \mathcal{A}'_c)^{\mathrm{GL}_n}) = P_{m,n} \cdot 1$.
  $\mathcal{P}$: Procesi bundle.

Conjecture: There exists a $\mathrm{GL}_n \times \mathbb{C}^* \times \mathbb{C}^*$-equivariant isomorphism: $Rp'_* \mathcal{A}'_c \cong q^{1-n} Rp_* \mathcal{A}_c$.

References

CEE09: Calaque-Enriquez-Etingof, Universal KZB equations...
FT11: Feigin-Tsymbaliuk, Equivariant K-theory of Hilbert schemes...
GN15: Gorsky-Neguț, Refined knot invariants and...
GORS14: Gorsky-Oblomkov-Rasmussen-Shende, Torus knots and...
Mel22: Mellit, Homology of torus knots
SV13: Schiffmann-Vasserot, The elliptic Hall algebra and...