Rational Cherednik Algebras and Torus Knot Invariants

Xinchun Ma

University of Chicago

xcma@uchicago.edu

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Representations of Rational Cherednik algebras

$\mathfrak{h} \subset \mathfrak{sl}_n$, $W = S_n \supset S$ reflections, $c \in \mathbb{C}$

Definition (using Dunkl embedding)

The rational Cherednik algebra $\mathrm{H}_c := \mathrm{H}_c(\mathfrak{h}, W)$ is a subalgebra of $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \ltimes W$ generated by $\mathfrak{h}^*$, W, and $y_i - y_{i+1}$, $i = 1, \ldots, n-1$, with

$$y_i := \frac{\partial}{\partial x_i} - c \sum_{s \in S} \frac{\langle \alpha_s, x_i \rangle}{\alpha_s}(1 - s).$$

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eg: $n = 2$, $s = (12)$, $(y_1 - y_2)\big((x_1 - x_2)^k\big) =$

$$\left(\frac{\partial}{\partial_{x_1}} - \frac{\partial}{\partial_{x_2}} - 2c\frac{1-s}{x_1 - x_2}\right)(x_1 - x_2)^k = \begin{cases} (2k - 4c)(x_1 - x_2)^{k-1} & \text{when } k \text{ is odd} \\ 2k(x_1 - x_2)^{k-1} & \text{when } k \text{ is even} \end{cases}$$

Finite-dimensional representations of H_c

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Theorem (Berest-Etingof-Ginzburg, 2003)

When $c = \frac{m}{n}$ for $m \geq 1$, $(m, n) = 1$, the only finite-dim irrep of H_c is L_c.
Only when $c = \frac{m}{n}$ for $(m, n) = 1$ does H_c have finite-dim reps.

Dunkl form

Fourier transform: $\Phi_c(x_i) = y_i, \quad \Phi_c(y_i) = -x_i, \quad \Phi_c(w) = w$

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e.g: $(y_1 - y_2)\big((x_1-x_2)^k\big) = (\frac{\partial}{\partial_{x_1}} - \frac{\partial}{\partial_{x_2}} - 2c\frac{1-s}{x_1-x_2})(x_1-x_2)^k =$
$\begin{cases} (2k-4c)(x_1-x_2)^{k-1} & \text{when } k \text{ is odd} \\ 2k(x_1-x_2)^{k-1} & \text{when } k \text{ is even} \end{cases}$

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$$\Rightarrow I_{\frac{m}{2}} = \begin{cases} ((x_1-x_2)^m), & \mathrm{L}_{\frac{m}{2}} = \mathbb{C}[x_1-x_2]/((x_1-x_2)^m) \text{ when } m \text{ is odd} \\ (0), & \mathrm{L}_{\frac{m}{2}} = \mathbb{C}[\mathfrak{h}] \text{ when } m \text{ is even} \end{cases}.$$

HOMFLY as a graded character

Theorem (Gorsky-Oblomkov-Rasmussen-Shende, 2014)

$$\mathrm{HOMFLY}_{a,q}(T_{m,n}) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \mathrm{ch}_q\big(\mathrm{Hom}_{S_n}(\wedge^i(\mathfrak{h}), \mathrm{L}_{\frac{m}{n}})\big).$$

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Conjecture (GORS, 2014)

There exists a filtration on L_c whose associated t-grading yields the refined identity

$$\mathrm{ch}_{a,q,t}(\mathrm{HHH}(T_{m,n})) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \mathrm{ch}_{q,t}\big(\mathrm{Hom}_{S_n}(\wedge^i(\mathfrak{h}), \mathrm{L}_{\frac{m}{n}})\big).$$

Proved cases: $m = nk + 1$, $a = 0$

Gordon-Stafford (2005): $(\mathrm{L}_{k+\frac{1}{n}})^{S_n} \cong \Gamma(\mathrm{Hilb}_0^n(\mathbb{C}^2), \mathcal{O}(k))$ (doubly graded)

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Mellit (2017) : $\mathrm{ch}_{a=0,q,t}(\mathrm{HHH}(T_{m,n})) = C_{m,n}(q,t)$.

Filtrations

- inductive filtration
- algebraic (Chern) filtration
- perverse filtration ($\sim$ compactified Jacobian of $y^m = x^n$)
- Hodge filtration ($\sim$ cuspidal mirabolic $\mathcal{D}$-module)
- ......

Inductive filtration

Definition

The inductive filtration F^{ind} is defined inductively such that
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$$\text{Flip: when } m, n > 1, \quad \mathrm{eL}_{\frac{m}{n}} \cong \mathrm{eL}_{\frac{n}{m}}$$

$$\text{Shift: when } c > 1, \quad \mathrm{L}_c \cong \mathrm{H}_c\mathrm{e}_- \otimes_{\mathrm{eH}_{c-1}\mathrm{e}} \mathrm{eL}_{c-1}$$

($\mathrm{e} = \frac{1}{n!}\sum_{w\in W} w$ such that $\mathrm{eL}_c \cong \mathrm{L}_c^W$;
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e.g: $\frac{13}{5} \succ \frac{3}{5} \succ \frac{5}{3} \succ \frac{2}{3} \succ \frac{3}{2} \succ \frac{1}{2}$

Algebraic filtration

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$$F^{\mathfrak{a}}_j \mathrm{L}_c = \Phi_c((\mathfrak{a}^{j+1})^{\perp_c})\beta_c$$
$$F^{\mathrm{alg}}_i \mathrm{L}_c = \sum_{2j+k\leq i} F^{\mathfrak{a}}_j \mathrm{L}_c(k)$$

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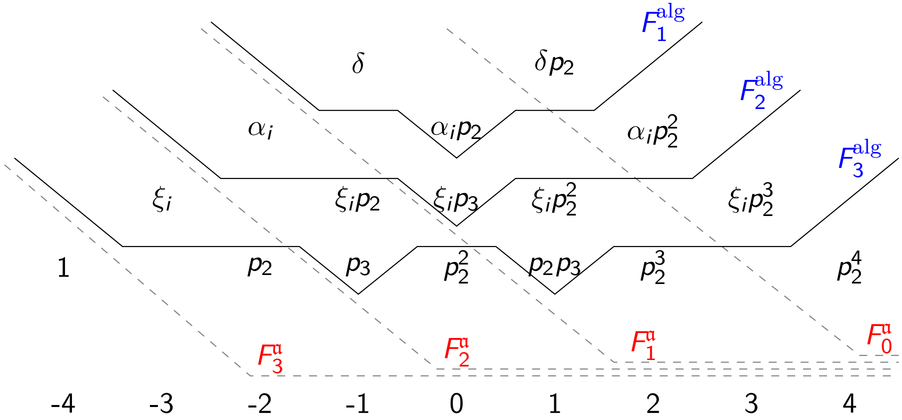
Theorem (M.)

$F^{\mathrm{alg}} = F^{\mathrm{ind}}$.

The case of $\mathfrak{sl}_3$ when $c = \frac{5}{3}$

$\mathfrak{a}^{\perp_c}$ = the space of c-harmonic polynomials spanned by
1, $\xi_i = x_i - x_{i+1}$, $\alpha_i = (y_i - y_{i+1})\delta$, $i = 1, 2$, δ (Vandermonte)



Proof strategy: two matrices

$\mathcal{R} := \mathbb{C}[\mathfrak{h}]/\mathfrak{a}$: coinvariant algebra

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$$\mathcal{R}^n \xrightarrow{A} \mathcal{R}^{n-k} \xrightarrow{B} \mathcal{R}^n \quad \text{for } 1 \leq k \leq n-1$$

$$A = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^k \\ & & \cdots & \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n^k \end{pmatrix} B = \begin{pmatrix} x_1^k & x_2^k & \cdots & x_n^k \\ & & \cdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

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$$F^{\mathrm{ind}} = F^{\mathrm{alg}} \text{ iff } \mathrm{Im}(A) = \ker(B) \qquad \text{for all } 1 \le k \le n-1$$

$$\text{iff } \mathrm{rank}(A) = \frac{(n-k)n!}{2} \qquad \text{for all } 1 \le k \le n-1.$$

Example: $k = n - 1$

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$$\mathcal{R}/\mathrm{Im}(A) = \mathbb{C}[\mathfrak{h}]/((\mathbb{C}[\mathfrak{h}]_+^W) + (x_1^{n-1}, \cdots, x_n^{n-1}))$$

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$$= \mathrm{H}^*(\mathcal{B}_{min}) = \mathrm{Ind}_{S_2}^{S_n}\mathrm{triv}$$

$\mathcal{B}_{min}$: Springer fiber at the minimal nilpotent orbit

Thank you!