Rational Cherednik Algebras and Torus Knot Homology

Xinchun Ma

University of Chicago

xcma@uchicago.edu

May 27, 2024

Khovanov-Rozansky homology

Defined by Khovanov using Soergel bimodules.

A triply graded generalization of $\underbrace{\text{Jones, Alexander,}}_{\text{1 variable}}\underbrace{\text{HOMFLY-PT}}_{\text{2 variables}}$ polynomials

Khovanov-Rozansky homology

Defined by Khovanov using Soergel bimodules.

A triply graded generalization of $\underbrace{\text{Jones, Alexander}}_{\text{1 variable}}$, $\underbrace{\text{HOMFLY-PT}}_{\text{2 variables}}$ polynomials

Example

$T_{2,3}$: $\text{HOMFLY}(a, q) = 1 + q + a$ (up to renormalization),

.

Khovanov-Rozansky homology

Defined by Khovanov using Soergel bimodules.

A triply graded generalization of $\underbrace{\text{Jones, Alexander}}_{\text{1 variable}}$, $\underbrace{\text{HOMFLY-PT}}_{\text{2 variables}}$ polynomials

Example

$T_{2,3}$: $\mathrm{HOMFLY}(a,q) = 1 + q + a$ (up to renormalization),

$T_{2,5}$: $\mathrm{HOMFLY}(a,q) = 1 + q + q^2 + a(1+q)$,

.

Khovanov-Rozansky homology

Defined by Khovanov using Soergel bimodules.

A triply graded generalization of $\underbrace{\text{Jones, Alexander}}_{\text{1 variable}}$, $\underbrace{\text{HOMFLY-PT}}_{\text{2 variables}}$ polynomials

Example

$T_{2,3}$: HOMFLY$(a, q) = 1 + q + a$ (up to renormalization),

$$\mathrm{ch}_{a,q,t}\mathrm{HHH}(T_{2,3}) = 1 + qt^{-1} + at^{-1}$$

$T_{2,5}$: HOMFLY$(a, q) = 1 + q + q^2 + a(1 + q)$,

.

Khovanov-Rozansky homology

Defined by Khovanov using Soergel bimodules.

A triply graded generalization of $\underbrace{\text{Jones, Alexander,}}_{\text{1 variable}}$ $\underbrace{\text{HOMFLY-PT}}_{\text{2 variables}}$ polynomials

Example

$T_{2,3}$: $\mathrm{HOMFLY}(a, q) = 1 + q + a$ (up to renormalization),

$$\mathrm{ch}_{a,q,t}\mathrm{HHH}(T_{2,3}) = 1 + qt^{-1} + at^{-1}$$

$T_{2,5}$: $\mathrm{HOMFLY}(a, q) = 1 + q + q^2 + a(1 + q)$,

$$\mathrm{ch}_{a,q,t}\mathrm{HHH}(T_{2,5}) = 1 + qt^{-1} + q^2t^{-2} + a(t^{-1} + qt^{-2}).$$

Representations of Rational Cherednik algebras

Representations of Rational Cherednik algebras

$\mathfrak{h} \subset \mathfrak{sl}_n$, $W = S_n \supset S$ reflections, $c \in \mathbb{C}$

Definition (using Dunkl embedding)

The rational Cherednik algebra $\mathrm{H}_c := \mathrm{H}_c(\mathfrak{h}, W)$ is a subalgebra of $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \ltimes W$ generated by $\mathfrak{h}^*$, W, and $y_i - y_{i+1}$, $i = 1, \ldots, n-1$, with

$$y_i := \frac{\partial}{\partial x_i} - c \sum_{s \in S} \frac{\langle \alpha_s, x_i \rangle}{\alpha_s}(1 - s).$$

Deformation of $\mathcal{D}(\mathfrak{h}) \ltimes W$ at c.

Finite-dimensional representations of H_c

Finite-dimensional representations of H_c

$\mathbb{C}[\mathfrak{h}]$: Polynomial representation of H_c.

$\mathbb{C}[\mathfrak{h}] \twoheadrightarrow \mathrm{L}_c$ maximal irreducible quotient

Finite-dimensional representations of H_c

$\mathbb{C}[\mathfrak{h}]$: Polynomial representation of H_c.

$\mathbb{C}[\mathfrak{h}] \twoheadrightarrow \mathrm{L}_c$ maximal irreducible quotient

Theorem (Berest-Etingof-Ginzburg, 2003)

When $c = \frac{m}{n}$ for $m \geq 1$, $(m, n) = 1$, the only finite-dim irrep of H_c is L_c.
Only when $c = \frac{m}{n}$ for $(m, n) = 1$ does H_c have finite-dim reps.

RCA vs KhR

Theorem (Gorsky-Oblomkov-Rasmussen-Shende, 2014)

$$\mathrm{HOMFLY}_{a,q}(T_{m,n}) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \mathrm{ch}_q\Big(\mathrm{Hom}_{S_n}(\wedge^i(\mathfrak{h}), \mathrm{L}_{\frac{m}{n}})\Big).$$

q: action of $\sum x_i y_i + y_i x_i$

RCA vs KhR

Theorem (Gorsky-Oblomkov-Rasmussen-Shende, 2014)

$$\mathrm{HOMFLY}_{a,q}(T_{m,n}) = a^{(n-1)(m-1)} \sum_{i=0}^{n-1} a^{2i} \mathrm{ch}_q\left(\mathrm{Hom}_{S_n}(\wedge^i(\mathfrak{h}), \mathrm{L}_{\frac{m}{n}})\right).$$

q: action of $\sum x_i y_i + y_i x_i$

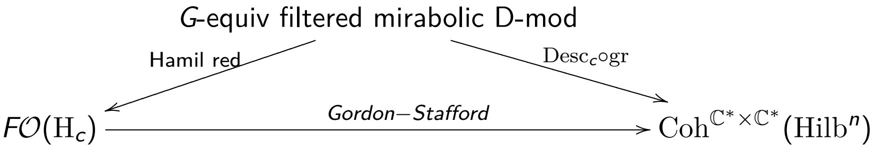
Conjecture (GORS, 2014), Theorem (M., 2024)

There exists a Hodge filtration on L_c whose associated t-grading yields the refined isomorphism

$$\mathrm{HHH}(T_{m,n}) \cong \mathrm{Hom}_{S_n}(\wedge^\bullet \mathfrak{h}, \mathrm{gr}_\bullet^{\mathrm{Hod}} \oplus \mathrm{L}_c(\bullet)).$$

Proof method

$F\mathcal{O}(\mathrm{H}_c)$: filtered objects in category $\mathcal{O}$ of H_c.



$\mathbf{N}_c$: cuspidal mirabolic D-module

- quantum Hamiltonian reduction of $\mathbf{N}_c$ equals $\mathrm{L}_c^{S_n}$.
- The equivariant K-theory class corresponding to $\mathrm{Desc}_c \circ \mathrm{gr}^{\mathrm{Hod}}(\mathbf{N}_c)$ has $\mathrm{HHH}(T_{m,n})$ as a matrix coefficient.

The example of $T(4,3)$, $a=0$

$$\overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{ccc}3&&\\1&2&4\end{array}}+\overbrace{\frac{qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q^2}\right)}}^{\begin{array}{ccc}2&&\\1&3&4\end{array}}$$

$$+\overbrace{\frac{(1-q)qt}{\left(1-\frac{q}{t}\right)\left(1-\frac{q^2}{t}\right)\left(1-\frac{t}{q}\right)^2}}^{\begin{array}{cc}3&4\\1&2\end{array}}+\overbrace{\frac{q^3}{\left(1-\frac{t}{q^3}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{ccc}4&&\\1&2&3\end{array}}+\overbrace{\frac{q^3}{\left(1-\frac{q^3}{t}\right)\left(1-\frac{t}{q^2}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{cccc}1&2&3&4\end{array}}$$

$$+\overbrace{\frac{qt(1-t)}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)^2\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{cc}4&\\2&\\1&3\end{array}}+\overbrace{\frac{qt}{\left(1-\frac{q^2}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{t}{q}\right)}}^{\begin{array}{cc}4&\\3&\\1&2\end{array}}$$

$$+\overbrace{\frac{qt(1-t)}{\left(1-\frac{q}{t}\right)^2\left(1-\frac{t}{q}\right)\left(1-\frac{t^2}{q}\right)}}^{\begin{array}{cc}2&4\\1&3\end{array}}+\overbrace{\frac{t^3}{\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)\left(1-\frac{q}{t^3}\right)}}^{\begin{array}{cc}3&\\2&\\1&4\end{array}}+\overbrace{\frac{t^3}{\left(1-\frac{q}{t^3}\right)\left(1-\frac{q}{t^2}\right)\left(1-\frac{q}{t}\right)}}^{\begin{array}{c}4\\3\\2\\1\end{array}}$$

$$=q^3+q^2t+qt+qt^2+t^3$$

q,t-Catalan number!

Thank you!